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Classical-Like Description of Quantum Dynamics by Means of Symplectic Tomography

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The dynamical equations of quantum mechanics are rewritten in the form of dynamical equations for the measurable, positive marginal distribution of the shifted, rotated, and squeezed quadrature introduced in the so-called "symplectic tomography." Then the possibility of a purely classical description of a quantum system as well as a reinterpretation of the quantum measurement theory is discussed and a comparision with the well-known quasi-probabilities approach is given. Furthermore, an analysis of the properties of this marginal distribution, which contains all the quantum information, is performed in the framework of classical probability theory. Finally, examples of the harmonic oscillator's states dynamics are treated.

1. INTRODUCTION

"Schrödinger made no secret of his intention to substitute simple classical pictures for the strange conceptions of quantum mechanics, for whose abstract character he expressed deep aversion." It is clear from this commentary of Rosenfeld[1] that from the early days of quantum theory there has been a permanent wish to understand quantum mechanics in terms of classical probabilities. However, due to the Heisenberg[2] and Schrödinger–Robertson[3,4] uncertainty relation for the position and momentum in quantum systems, the joint distribution function in the phase space does not exist. This leads to the introduction of the so-called quasi-probability distributions, such as the Wigner function,[5] the Husimi Q-function,[6]

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and the Glauber–Sudarshan P-function,[7,8] later on unified into a one-parametric family.[9] Furthermore, in order to bridge quantum and classical physics, Madelung[10] already observed that the modulus and the phase of the wave function obey the hydrodynamical classical equations, and along this line the stochastic quantization scheme has been suggested by Nelson[11] to link the classical stochastic mechanics formalism with the quantum mechanical basic entities, such as wave function and propagator. In some sense, also the hidden variables[12] was proposed to relate the quantum processes to the classical ones. Nevertheless, to date there does not exist a formalism which consistently connects the "two worlds."

The discussed quasi-probabilities illuminated the similarities and the differences between classical and quantum considerations and they are widely used as instruments for calculations in quantum theory.[13,14] However, they cannot play the role of classical distributions since, for example, the Wigner function and the P-function may have negative values. Althought the Q-function is always positive and normalized, it does not describe measurable distributions of concrete physical variables.

Recently, after J. Bertrand and P. Bertrand[15] made the first attempt to apply the tomographic principle to phase space distributions, Vogel and Risken,[16] using the formalism of Ref. 9, established an integral relation between the Wigner function and the marginal distribution for the measurable homodyne output variable which represents a rotated quadrature of the electromagnetic field. This result gives the possibility of "measuring" the quantum state, and it is referred to as optical homodyne tomography.[17]

In Ref. 18 a symplectic tomography procedure was suggested to obtain the Wigner function by measuring the marginal distribution for a shifted, rotated, and squeezed quadrature, which depends on extra parameters. In Ref. 19 the formalism of Ref. 16 was formulated in invariant form, relating the homodyne output distribution directly to the density operator. Then, in Ref. 20 the symplectic tomography formalism was also formulated in this invariant form and was extended to the multimode case. Thus, due to the introduction of the quantum tomography procedure, the real positive marginal distribution for measurable observables, such as rotated shifted and squeezed quadratures, turned out to determine completely the quantum states.

The aim of the present work is to formulate the standard quantum dynamics in terms of the classical marginal distribution of the measurable shifted, rotated, and squeezed quadrature components, used in the symplectic tomography scheme. Thus, we obtain an alternative formulation of the quantum system evolution in terms of the evolution of a real and positive distribution function for measurable physical observables. We will show the connection of such a "classical" probability evolution with the evolution of the above-discussed quasi-probability distributions. Preliminarily, the approach was briefly presented in Ref. 21.

Examples relative to the states of a harmonic oscillator and free motion will be considered in the framework of the given formulation of quantum mechanics as well as an oscillator with friction and driven terms included.

2. DENSITY OPERATOR AND DISTRIBUTION FOR SHIFTED ROTATED AND SQUEEZED QUADRATURE

In Ref. 18 an operator $\hat{X}$ was introduced as the generic linear combination of the position $\hat{q}$ and momentum $\hat{p}(\hbar=1)$

$$\hat{X}=\mu\hat{q}+\nu\hat{p}+\delta \tag{1}$$

which depends upon three real parameters μ, ν, δ and, due to its hermiticity, is a measurable observable. Thus, the marginal distribution, defined as the Fourier transform of the characteristic function

$$w(X, \mu, \nu, \delta)=\int dk e^{-ikX}\langle e^{ik\hat{X}}\rangle \tag{2}$$

depends on the parameters μ, ν, δ, and it is normalized with respect to the X variable

$$\int dX w(X, \mu, \nu, \delta)=1 \tag{3}$$

Furthermore, it was shown[18] that this marginal distribution is related to the state of the quantum system, expressed in terms of its Wigner function $W(q, p)$, as follows:

$$w(X, \mu, \nu, \delta)=\int e^{-ik(X-\mu q-\nu p-\delta)}W(q, p)\frac{dk\, dq\, dp}{(2\pi)^2} \tag{4}$$

Equation (4) shows that w is a function of the difference $X-\delta=x$, so that it can be rewritten as

$$w(x, \mu, \nu)=\int e^{-ik(x-\mu q-\nu p)}W(p, q)\frac{dk\, dq\, dp}{(2\pi)^2} \tag{5}$$

This formula can be inverted and the Wigner function of the state can be expressed in terms of the marginal distribution[18]

$$W(q, p) = (2\pi)^2 z^2 w_F(z, -zq, -zp) \tag{6}$$

where $w_F(z, a, b)$ is the Fourier component of the marginal distribution (5) taken with respect to the variables x, μ, ν, i.e.,

$$w_F(z, a, b) = \frac{1}{(2\pi)^3} \int w(x, \mu, \nu)\, e^{-i(xz + \mu a + \nu b)}\, dx\, d\mu\, d\nu \tag{7}$$

Hence, it was shown that the quantum state could be described by the positive classical marginal distribution for the squeezed rotated, and shifted quadratures which could be considered as a classical probability associated to a stochastic variable x and depending also on the parameters.

In the case of only rotated quadrature, $\mu = \cos\phi$ and $\nu = \sin\phi$, the usual optical tomography formula of Ref. 16 gives the same possibility through the Radon transform instead of the Fourier transform. This is, in fact, a partial case of the symplectic transformation of quadrature since the rotation group is a subgroup of the symplectic group $ISp(2, R)$ whose parameters are used to describe the transformation (1).

In Ref. 20 an invariant form connecting directly the marginal distribution $w(x, \mu, \nu)$ and the density operator was found:

$$\hat{\rho} = \int dx\, d\mu\, d\nu w(x, \mu, \nu)\, \hat{K}_{\mu,\nu} \tag{8}$$

where the kernel operator has the form

$$\hat{K}_{\mu,\nu} = \frac{1}{2\pi} z^2 e^{-izx} e^{-iz^2\mu\nu/2} e^{iz\nu\hat{p}} e^{iz\mu\hat{q}} \tag{9}$$

Formulas (6) and (8) of symplectic tomography show that there exists an invertible map between the quantum states described by the set of non-negative and normalized hermitian density operators $\hat{\rho}$ and the set of positive, normalized marginal distributions ("classical" ones) for the measurable shifted, rotated, and squeezed quadratures. So, the information contained in the marginal distribution is the same as that contained in the density operator; and due to this, one could represent the quantum dynamics in terms of evolution of the marginal probability. Indeed, the fact that $\hat{K}_{\mu,\nu}$ depends also on the z variable (i.e., each Fourier component gives a self-consistent kernel) shows the overcompleteness of information achievable by measuring the observable of Eq. (1).

The definition of the marginal distribution function $w(x, \mu, \nu)$ might be alternatively given in terms of the eigenstates of the operator $\hat{x}=\hat{X}-\delta$

$$\hat{x}\,|x\rangle = x\,|x\rangle \tag{10}$$

which can be obtained from the position eigenstates

$$\hat{q}\,|q\rangle = q\,|q\rangle \tag{11}$$

by the action of the unitary operator $\hat{S}$

$$|x\rangle = \hat{S}\,|q\rangle \tag{12}$$

which represents the composition of simple operations such as rotation and squeezing, i.e., it satisfies the requierement

$$\hat{S}^{\dagger}\hat{q}\hat{S} = \mu\hat{q} + \nu\hat{p} \tag{13}$$

It is worth remarking about this transformation, that there exist a constraint[22] due to the commutation relation between the observable (1) and its canonical conjugate, i.e., if one introduce the observable

$$P = \mu'\hat{q} + \nu'\hat{p} + \delta' \tag{14}$$

the matrix

$$\Lambda = \begin{pmatrix} \mu & \nu \\ \mu' & \nu' \end{pmatrix} \tag{15}$$

must satisfy the relation

$$\Lambda\sigma\Lambda^{T} = \sigma, \qquad \sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{16}$$

Then, the marginal distribution is the diagonal matrix element of the density operator in the transformed basis (10)

$$w(x, \mu, \nu) = \langle x|\,\hat{\rho}\,|x\rangle = \mathrm{Tr}\{\hat{\rho}\,|x\rangle\langle x|\} \tag{17}$$

or it is the diagonal matrix element in positional representation of the transformed density operator

$$w(x, \mu, \nu) = \langle q|\,\hat{S}^{\dagger}\hat{\rho}\hat{S}\,|q\rangle = \mathrm{Tr}\{\hat{S}^{\dagger}\hat{\rho}\hat{S}\,|q\rangle\langle q|\} \tag{18}$$

The form of the shifted and squeezed operator $\hat{S}$ is well known.[23] Choosing the parameters $\mu=\cos\phi$ and $\nu=\sin\phi$, the operator $\hat{S}$ gives the marginal distribution for the homodyne output of Ref. 16. In the case of $\mu=1$ and $\nu=0$ the marginal distribution is that for quadrature $\hat{q}$, i.e., $w(q,1,0)=\rho(q,q)=\langle q|\,\hat{\rho}\,|q\rangle$, while in the case of $\mu=0$ and $\nu=1$ the marginal distribution is that for the other quadrature $\hat{p}$, i.e., $w(p,0,1)=\rho(p,p)=\langle p|\,\hat{\rho}\,|p\rangle$.

3. QUANTUM EVOLUTION AS A CLASSICAL PROCESS

We now derive the evolution equation for the marginal distribution function w using the invariant form of the connection between the marginal distribution and the density operator given by the formula (8). Then, from the equation of motion for the density operator which includes the interaction with environment $\chi(\rho)$

$$\partial_t\hat{\rho}=-i[\hat{H},\hat{\rho}]+\chi(\rho) \tag{19}$$

we obtain the evolution equation for the marginal distribution in the form

$$\begin{aligned}\int dx\,d\mu\,d\nu\{\dot{w}(x,\mu,\nu,t)\,\hat{K}_{\mu,\nu}+w(x,\mu,\nu,t)\,\hat{I}_{\mu,\nu}\}\\ =\chi\left(\int dx\,d\mu\,d\nu w(x,\mu,\nu,t)\,\hat{K}_{\mu,\nu}\right)\end{aligned} \tag{20}$$

in which the known Hamiltonian determines the kernel $\hat{I}_{\mu,\nu}$ through the commutator

$$\hat{I}_{\mu,\nu}=i[\hat{H},\hat{K}_{\mu,\nu}] \tag{21}$$

while the r.h.s. is functionally dependent on the marginal distribution. The obtained integral-operator equation can be reduced to an integro-differential equation for the function w in some cases. Let us consider first the situation in which $\chi(\rho)=0$; the opposite situation will be discussed later. Then we represent the kernel operator $\hat{I}_{\mu,\nu}$ in normal order form (i.e., all the momentum operators on the left side and the position ones on the right side) containing the operator $\hat{K}_{\mu,\nu}$ as follows:

$$:\hat{I}_{\mu,\nu}:=\mathscr{R}(\hat{p}):\hat{K}_{\mu,\nu}:\mathscr{P}(\hat{q}) \tag{22}$$

where $\mathscr{R}(\hat{p})$ and $\mathscr{P}(\hat{q})$ are finite or infinite operator polynomials (depending also on the parameters μ and ν) determined by the Hamiltonian. Then we calculate the matrix elements of the operator equation (20) between the states $\langle q|$ and $|q\rangle$, obtaining

$$\int dx\, d\mu\, d\nu \{ \dot{w}(x, \mu, \nu, t) + w(x, \mu, \nu, t)\, \mathscr{R}(p)\, \mathscr{P}(q) \} \langle p| : \hat{K}_{\mu,\nu} \colon |q\rangle = 0 \quad (23)$$

If we write

$$\mathscr{R}(p)\, \mathscr{P}(q) = \Pi(p, q) = \sum_n \sum_m c_{n,m}(z, \mu, \nu)\, p^n q^m \quad (24)$$

due to the particular form of the kernel in Eq. (9), Eq. (23) can be rewritten as

$$\int dx\, d\mu\, d\nu \{ \dot{w}(x, \mu, \nu, t) + w(x, \mu, \nu, t)\, \vec{\Pi}(\tilde{p}, \tilde{q}) \} \langle p| : \hat{K}_{\mu,\nu} \colon |q\rangle = 0 \quad (25)$$

where $\tilde{p}$, $\tilde{q}$ are operators of the form

$$\tilde{p} = \left(-\frac{i}{z}\frac{\partial}{\partial \nu} + \frac{\mu}{2} z \right), \qquad \tilde{q} = \left(-\frac{i}{z}\frac{\partial}{\partial \mu} + \frac{\nu}{2} z \right) \quad (26)$$

while z, in the space of variables x, μ, ν, should be understood as the derivative with respect to x, i.e.,

$$z \leftrightarrow i \frac{\partial}{\partial x} \quad (27)$$

and when it appears in the denominator, it is understood as an integral operator. Furthermore, the right arrow over Π means that, with respect to the order of Eq. (24), the operators $\tilde{p}$ and $\tilde{q}$ act on the right, i.e., on $\langle p| : \hat{K}_{\mu,\nu} \colon |q\rangle$. Under the hypothesis of regularity of w on the boundaries, we can perform integration by parts in Eq. (25) disregarding the surface terms, to get

$$\int dx\, d\mu\, d\nu \{ \dot{w}(x, \mu, \nu, t) + w(x, \mu, \nu, t)\, \overleftarrow{\Pi}(\check{p}, \check{q}) \} \langle p| : \hat{K}_{\mu,\nu} \colon |q\rangle = 0 \quad (28)$$

where now $\overleftarrow{\Pi}$ means that the operators $\check{p}$, $\check{q}$

$$\check{p} = \left(-\frac{i}{z}\frac{\partial}{\partial \nu} - \frac{\mu}{2} z \right), \qquad \check{q} = \left(-\frac{i}{z}\frac{\partial}{\partial \mu} - \frac{\nu}{2} z \right) \quad (29)$$

act on the left, i.e., on the product of coefficients $c_{n,m}(-z, \mu, \nu)$ with the marginal distribution w. Finally, using the completeness property of the Fourier exponents given by $\langle p| : \hat{K}_{\mu,\nu} : |q\rangle$, we arrive at the following equation of motion for the marginal distribution function:

$$\partial_t w + w\overleftarrow{\Pi}(\check{p}, \check{q}) = 0 \tag{30}$$

Let us consider the important example of the motion of the particle in a potential with the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2} + V(\hat{q}) \tag{31}$$

then the described procedure of calculating the normal-order kernel (22) gives the following form of the quantum dynamics in terms of a Fokker–Planck-like equation for the marginal distribution

$$\dot{w} - \mu \frac{\partial}{\partial \nu} w - i \left[V\left(\frac{-1}{\partial/\partial x} \frac{\partial}{\partial \mu} - i \frac{\nu}{2} \frac{\partial}{\partial x} \right) - V\left(\frac{-1}{\partial/\partial x} \frac{\partial}{\partial \mu} + i \frac{\nu}{2} \frac{\partial}{\partial x} \right) \right] w = 0 \tag{32}$$

which in the general case is an integro-differential equation. It is worth remarking that, considering the quadrature X of Eq. (1) to be dimensionless, the Planck constant $\hbar$ should appear in Eq. (32) to multiply the first two terms. As a consequence it is clear that the equation, even if classical-like, gives a quantum description of the system evolution (as the Schrödinger equation).

Thus, given a Hamiltonian of the form (31) we can study the quantum evolution of the system writing down a Fokker–Planck-like equation for the marginal distribution. Solving this one for a given initial positive and normalized marginal distribution we can obtain the quantum density operator $\hat{\rho}(t)$ according to Eq. (8). Conceptually this means that we can discuss the system quantum evolution considering classical, real, positive, and normalized distributions for the measurable variable X which is a shifted, rotated, and squeezed quadrature. The distribution function which depends on extra parameters obeys a classical equation which preserves the normalization condition of the distribution. In this sense we can always reduce the quantum behavior of the system to the classical behavior of the marginal distribution. Of course, this statement respects the uncertainty relation because the measurable marginal distribution is the distribution for one observable. That is the essential difference (despite some similarities) between the introduced marginal distribution and the discussed quasi-distributions, including the real positive Q-function, which depend on the two variables of the phase space and are normalized with respect to these variables. We should point out that we do not derive quantum mechanics from classical stochastic mechanics, i.e., we do not quantize any classical stochastic process; our result is to present the quantum dynamics equations as classical ones, and in doing this we need not only the classical Hamiltonian but also its quantum counterpart.

4. EXAMPLES

Let us choose, as the system to study, a driven harmonic oscillator of unit mass with a hamiltonian of the type

$$H=\frac{p^2}{2}+\omega^2\frac{q^2}{2}-fq \tag{33}$$

then from Eq. (32) immediately follows

$$\dot{w}-\mu\frac{\partial}{\partial\nu}w+\omega^2\nu\frac{\partial}{\partial\mu}w+f\nu\frac{\partial}{\partial x}w=0 \tag{34}$$

Below we cosider solutions of some special cases of Eq. (34), while the solution for the complete equation will be given in the next section by using a propagator method.

4.1. Free Motion

For the free motion, $\omega=f=0$, the evolution equation (34) becomes the first-order partial differential equation

$$\dot{w}-\mu\frac{\partial}{\partial\nu}w=0 \tag{35}$$

and it has a gaussian solution of the form

$$w(x,\mu,\nu,t)=\frac{1}{\sqrt{2\pi\sigma_x(t)}}\exp\left\{-\frac{x^2}{2\sigma_x(t)}\right\} \tag{36}$$

where the dispersion of the observable $\hat{x}$ depends on time and parameters as follow:

$$\sigma_x(t)=\tfrac{1}{2}[\mu^2(1+t^2)+\nu^2+2\mu\nu t] \tag{37}$$

The initial condition corresponds to the marginal distribution of the ground state of an artificial harmonic oscillator calculated from the respective Wigner function.[18]

4.2. Harmonic Oscillator

For the simple harmonic oscillator with frequency $\omega = 1$, we have $f = 0$; then Eq. (34) becomes

$$\dot{w} - \mu \frac{\partial}{\partial \nu} w + \nu \frac{\partial}{\partial \mu} w = 0 \tag{38}$$

If we consider the first excited state of the harmonic oscillator, we know the Wigner function[24]

$$W_1(q, p) = -2(1 - 2q^2 - 2p^2) \exp[-q^2 - p^2] \tag{39}$$

The result is time independent due to the stationarity of the state, but for small q and p it becomes negative while the solution of Eq. (38)

$$w_1(x, \mu, \nu, t) = \frac{2}{\sqrt{\pi}} [\mu^2 + \nu^2]^{-3/2} x^2 \exp\left\{-\frac{x^2}{\mu^2 + \nu^2}\right\} \tag{40}$$

is itself time independent but everywhere positive.

Indeed, a time evolution is present explicitly in the coherent state, whose Wigner function is given by

$$\begin{aligned} W_c(q, p) = 2 \exp\{&-q^2 - q_0^2 - p^2 - p_0^2 \\ &+ 2(qq_0 + pp_0) \cos t - (pq_0 - qp_0) \sin t\} \end{aligned} \tag{41}$$

where q_0 and p_0 are the initial values of position and momentum. For the same state the marginal distribution shows a more complicated evolution:

$$\begin{aligned} w_c(x, \mu, \nu, t) = &\frac{1}{\sqrt{\pi}} [\mu^2 + \nu^2]^{-1/2} \\ &\times \exp\left\{-q_0^2 - p_0^2 - \frac{x^2}{\nu^2} + 2\frac{x}{\nu}(p_0 \cos t - q_0 \sin t)\right\} \\ &\times \exp\left\{\frac{1}{\mu^2 + \nu^2}\left[\frac{\mu}{\nu} x + q_0(\mu \sin t + \nu \cos t) \right.\right. \\ &\left.\left. + p_0(\nu \sin t - \mu \cos t)\right]^2\right\} \end{aligned} \tag{42}$$

It is also interesting to consider the comparison between the Wigner function and the marginal probability for nonclassical states of the harmonic oscillator, such as the female cat state defined as[25]

$$|\alpha_-\rangle = N_-(|\alpha\rangle - |-\alpha\rangle), \qquad \alpha = 2^{-1/2}(q_0 + ip_0) \tag{43}$$

with

$$N_- = \left\{\frac{\exp[(q_0^2 + p_0^2)/2]}{4\sinh[(q_0^2 + p_0^2)/2]}\right\}^{1/2} \tag{44}$$

and for which the Wigner function assumes the following form:

$$\begin{aligned} W_-(q,p) = 2N_-^2 e^{-q^2-p^2}\{e^{-q_0^2-p_0^2}&\cosh[2(qq_0+pp_0)\cos t + 2(qp_0 - pq_0)\sin t] \\ &- \cos[2(qp_0 - pq_0)\cos t - 2(qq_0 + pp_0)\sin t]\} \end{aligned} \tag{45}$$

The corresponding marginal distribution is

$$\begin{aligned} w_-(x,\mu,\nu,t) = N_-^2 (w_A(x,\mu,\nu,t) &- w_B(x,\mu,\nu,t) \\ &- w_B^*(x,\mu,\nu,t) + w_A(-x,\mu,\nu,t)] \end{aligned} \tag{46}$$

with

$$\begin{aligned} w_A(x,\mu,\nu,t) = \frac{1}{\sqrt{\pi}}&[\mu^2+\nu^2]^{-1/2} \\ &\times \exp\left\{-q_0^2 - p_0^2 - \frac{x^2}{\nu^2} + 2\frac{x}{\nu}(p_0\cos t - q_0\sin t)\right\} \\ &\times \exp\left\{\frac{1}{\mu^2+\nu^2}\left[\frac{\mu}{\nu}x + q_0(\mu\sin t + \nu\cos t)\right.\right. \\ &\left.\left.- p_0(\nu\sin t - \mu\cos t)\right]^2\right\} \end{aligned} \tag{47}$$

and

$$\begin{aligned} w_B(x,\mu,\nu,t) = \frac{1}{\sqrt{\pi}}[\mu^2+\nu^2]^{-1/2} &\exp\left\{-\frac{x^2}{\nu^2} - 2i\frac{x}{\nu}(q_0\cos t + p_0\sin t)\right\} \\ \times \exp&\left\{\frac{-1}{\mu^2+\nu^2}\left[-i\frac{\mu}{\nu}x + q_0(\mu\cos t - \nu\sin t)\right.\right. \\ &\left.\left.+ p_0(\mu\sin t + \nu\cos t)\right]^2\right\} \end{aligned} \tag{48}$$

The presented examples show that, for the evolution of the state of a quantum system, one can always associate the evolution of the probability density with the random classical variable X which obeys "classical" Fokker–Planck-like equations, and this probability density contains the same information (about a quantum system) which is contained in any quasi-distribution function. But the probability density has the advantage that it behaves completely as the usual classical one. The physical meaning of the "classical" random variable X is transparent; it is considered as the position in an ensemble of shifted, rotated, and scaled rest frames in the classical phase space of the system under study. We note that for non-normalized quantum states, such as the states with fixed momentum (De Broglie wave) or with fixed position, the introduced map in Eq. (8) may be preserved. In this context the plane wave states of free motion have the marginal distribution corresponding to the classical white noise, as we shall see below.

4.3. Squeezed Coherent Sates

Here we will consider the marginal distribution $w(x, \mu, \nu)$ for the squeezed coherent states of the harmonic oscillator. The Wigner function of these pure Gaussian states may be represented in the form[26]

$$W_\alpha(q, p) = 2 \exp\left[-\frac{1}{2}(p-\bar{p}(t), q-\bar{q}(t))\, \mathbf{m}^{-1} \begin{pmatrix} p-\bar{p}(t) \\ q-\bar{q}(t) \end{pmatrix} \right] \tag{49}$$

where $\mathbf{m}$ is the dispersion matrix

$$\mathbf{m} = \begin{pmatrix} \sigma_p(t) & \sigma_{pq}(t) \\ \sigma_{pq}(t) & \sigma_q(t) \end{pmatrix} \tag{50}$$

and $\bar{p}(t)$ and $\bar{q}(t)$ are the mean values of the quadratures

$$\bar{q}(t) = \sqrt{2}\, \Re(\alpha e^{-it}), \qquad \bar{p}(t) = \sqrt{2}\, \Im(\alpha e^{-it}) \tag{51}$$

The variances in Eq. (50) are given by

$$\sigma_p(t) = \frac{1}{2}\left(s \cos^2 t + \frac{1}{s} \sin^2 t \right) \tag{52}$$

$$\sigma_q(t) = \frac{1}{2}\left(\frac{1}{s} \cos^2 t + s \sin^2 t \right) \tag{53}$$

$$\sigma_{pq}(t) = \frac{1}{2}\left(s - \frac{1}{s} \right) \sin t \cos t \tag{54}$$

with s the squeezing parameter. Using Eq. (49) in formulas (6) and (7), we obtain for the marginal distribution the expression

$$w(x, \mu, v, t) = \frac{1}{\sqrt{2\pi\sigma_x(t)}} \exp\left\{-\frac{[x - \mu\bar{q}(t) - v\bar{p}(t)]^2}{2\sigma_x(t)}\right\} \tag{55}$$

where

$$\sigma_x(t) = \tfrac{1}{2}[\mu^2\sigma_q(t) + v^2\sigma_p(t) + 2\mu v\sigma_{pq}(t)] \tag{56}$$

Let us now take the limit $s \to 0$; this means that our marginal distribution becomes a delta function

$$\lim_{s \to 0} w = \delta(x - (\mu q_0 + v p_0) \cos t - (\mu p_0 - v q_0) \sin t) \tag{57}$$

and as a consequence its spectrum will be constant and equal to unity for each value of the variable conjugate to X; thus, it will correspond to the white noise spectrum. On the other hand, the nonnormalized quantum states, such as the states with fixed momentum (De Broglie wave) or with fixed position, have a marginal distribution normalized and everywhere equal to one. Thus, plane wave states of free motion correspond to the classical white noise distribution.

5. EVOLUTION IN THE PRESENCE OF ENVIRONMENTAL INTERACTION

When a system is coupled with the "rest of the universe" the time evolution of the density operator is no longer unitary, and to treat the problem at the quantum level, one needs some approximations; usually the starting point is a simple system such as a harmonic oscillator that interacts linearly with a bath idealized as an infinity of other harmonic oscillators; then the (master) equation for the density operator becomes[27]

$$\dot{\rho} = -i[a^\dagger a, \rho] + \chi(\rho)$$

$$\chi(\rho) = \frac{\gamma}{2}(\bar{n} + 1)(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a)$$

$$+ \frac{\gamma}{2}\bar{n}(2a^\dagger\rho a - aa^\dagger\rho - \rho aa^\dagger) \tag{58}$$

where γ is the damping constant characterizing the relaxation time of the system, $a, a^\dagger$ are the boson operators of the system, and $\bar{n}$ is the number of the thermal excitation of the bath. Using Eq. (19) in the interaction picture and performing step-by-step the same procedure that leads to Eq. (30), one may describe the damped evolution by means of

$$\dot{w} = \frac{\gamma}{2}\left[2 - \frac{\partial}{\partial \nu}\nu - \frac{\partial}{\partial \mu}\mu + \frac{1}{2}(\mu^2 + \nu^2)\frac{\partial^2}{\partial x^2}\right] w \tag{59}$$

where we have assumed for simplicity $\bar{n} = 0$, a situation common in quantum optical systems. In Eq. (59) we recognize the Fokker–Planck equation, where the diffusion term is given by the proper stochastic term while the drift is given by the parameters (the factor 2 can be eliminated by a simple transformation $w = \tilde{w}e^{\gamma t}$). The solution of Eq. (59), with coherent initial excitation q_0, p_0, is

$$w(x, \mu, \nu, t) = \frac{1}{\pi}\frac{1}{\sqrt{\pi(\mu^2+\nu^2)}}\exp\left\{-\frac{[x + (\mu q_0 - \nu p_0)\, e^{-\gamma t/2}]^2}{\mu^2+\nu^2}\right\} \tag{60}$$

which is exactely the Fourier transform of the Wigner function for the damped harmonic oscillator given by[24]

$$W(q, p) = 2\exp[-(q - q_0 e^{-\gamma t/2})^2 - (p - p_0 e^{-\gamma t/2})^2] \tag{61}$$

This is proof that the developed formalism is consistent also in the case of open quantum systems.

6. QUANTUM MEASUREMENTS AND CLASSICAL MEASUREMENTS

In this section we will discuss the concept of quantum measurements in the framework of the developed approach. It is a well-known statement[28, 29] that quantum mechanics suffers from an inconsistency in the sense that it needs, for its understanding, a classical device measuring quantum observables. Due to this, the theory of measurements supposes that there exist two worlds: the classical one and the quantum one. Of course in the classical world the measurements of classical observables are produced by classical devices. In the quantum world the measurements of quantum observables are produced by classical devices too. Due to this, the theory of quantum measurements is considered as something very specifically different from the classical measurements.

Recently there have been proposed some schemes[30,31] to resolve the dichotomy between the measured microsystem and the measuring macro-apparatus; however, it is psycologically accepted that to understand the physical meaning of a measurement in the classical world is much easier than to understand the physical meaning of an analogous measurement in the quantum world.

Our aim is to show that in fact all the roots of difficulties of quantum measurements are present in the classical measurements as well. Using the invertible map of the quantum states (both normalized and nonnormalized) and classical states (described by classical distributions and generalized functions) given by Eq. (8), we can conclude that the complete information about a quantum state is obtained from purely classical measurements of the position of a particle, made by classical devices in each reference frame of the ensemble of the classical reference frames, which are shifted, scaled, and rotated in the classical phase space.

These measurements do not need any quantum language, if we know how to produce, in the classical world (using the notion of classical position and momentum), reference frames in the classical phase space differing from each other by rotation, scaling, and shifting of the axis of the reference frame and how to measure only the position of the particle from the point of view of these different reference frames. Thus, knowing how to obtain the classical marginal distribution function $w(x, \mu, \nu)$, which depends on the parameters μ, ν, δ, labeling each reference frame in the classical phase space, we reconstruct through the map (8) the quantum density operator.

By this approach, we avoid the unpleasant paradox of the quantum world which needs, for its explanation, measurements by a classical apparatus. Nevertheless, all the difficulties of the quantum approach continue to be present, but in a different classical form. In fact, if we consider, for example, the notion of wave function collapse,[32] it is displaced in the classical framework, since if we idealize the measuring apparatus as a bath with which the system interacts,[33] then a reduction of the probability distribution (as our marginal distribution) occurs as soon as we "pick" a value (hence a trajectory) of the classical stochastic process associated to the observable (as that of Eq. (1)).

About the developed formalism we are aware that the crucial point might be the practical realization of the generic linear quadratures such as in Eq. (1). Then, let us consider a practical implementation, in the optical domain. The quadrature of Eq. (1) could be experimentally accessible by using, for example, the squeezing pre-amplification (pre-attenuation) of a field mode that is going to be measured (a similar method in a different context was discussed in Ref. 34). In fact, let $\hat{a}$ be the signal field mode to be detected; when it passes through a squeezer, it becomes $\hat{a}_s = \hat{a}\cosh s - \hat{a}^\dagger e^{i\theta}\sinh s$, where s and θ characterize the complex squeezing parameter $\zeta = se^{i\theta}$.[35] Then, if we subsequently detect the field by using the balanced homodyne scheme, we get an output signal proportional to the average of the following quadrature:

$$\hat{E}(\phi) = \frac{1}{\sqrt{2}}(\hat{a}_s e^{-i\phi} + \hat{a}_s^\dagger e^{i\phi}) \tag{62}$$

where ϕ is the local oscillator phase. When this phase is locked to that of the squeezer, such that $\phi = \theta/2$, Eq. (62) becomes

$$\hat{E}(\phi) = \frac{1}{\sqrt{2}}(\hat{a}e^{-i\theta/2}[\cosh s - \sinh s] + \hat{a}^\dagger e^{i\theta/2}[\cosh s - \sinh s]) \tag{63}$$

which, essentially, coincides with Eq. (1), if one recognizes the independent parameters

$$\mu = [\cosh s - \sinh s]\cos(\theta/2), \qquad \nu = [\cosh s - \sinh s]\sin(\theta/2) \tag{64}$$

The shift parameter δ does not have a real physical meaning, since it causes only a displacement of the distribution along the X line without changing its shape, as can be seen from Eqs. (4) and (5). So, in a practical situation, it can be omitted. To be more precise, the shift parameter does not play a real physical role in the measurement process; it has been introduced for formal completeness and expresses the possibility to achieve the desired marginal distribution by performing the measurements in an ensemble of frames that are shifted from each other (a related method was discussed early in Ref. 36). In an electro-optical system this only means having the freedom to use different photocurrent scales in which the zero is shifted by a known amount.

7. CONNECTION WITH MEASUREMENTS IN HOMODYNE TOMOGRAPHY

At this point, a comparison with the usual tomographic technique, used in experiments of the type of Ref. 17, is useful. To this end, we recall that in this case the timelike evolution of the system is brought about by changing the parameters, and thus no explicit time dependency of w is needed. Furthermore, we note that a relation between the density operator and the marginal distribution analogous to Eq. (8) can be derived starting from another operator identity such as[9]

$$\hat{\rho} = \int \frac{d^2\alpha}{\pi} \operatorname{Tr}\{\hat{\rho}\hat{D}(\alpha)\} D^{-1}(\alpha) \tag{65}$$

which, by the change of variables $\mu = -\sqrt{2}\,\Im\alpha$, $\nu = \sqrt{2}\,\Re\alpha$, becomes

$$\hat{\rho} = \frac{1}{2\pi} \int d\mu\, d\nu \operatorname{Tr}\{\hat{\rho}e^{-i\hat{X}}\}\, e^{i\hat{X}} = \frac{1}{2\pi} \int d\mu\, d\nu \operatorname{Tr}\{\hat{\rho}e^{-i\hat{x}}\}\, e^{i\hat{x}} \tag{66}$$

The trace can now be evaluated using the complete set of eigenvectors $\{|x\rangle\}$ for the operator $\hat{x}$; we obtain

$$\operatorname{Tr}\{\hat{\rho}e^{-i\hat{x}}\} = \int dx w(x, \mu, \nu)\, e^{-ix} \tag{67}$$

Inserting this into Eq. (66), we have a relation of the same form as Eq. (8) with the kernel given by

$$\hat{K}_{\mu,\nu} = \frac{1}{2\pi} e^{-ix} e^{i\hat{x}} = \frac{1}{2\pi} e^{-ix} e^{i\mu\hat{q} + i\nu\hat{p}}, \tag{68}$$

which is the same as Eq. (9) setting $z = 1$. This means that we now have only one particular Fourier component due to the particular change of variables (the most general should be $z\mu = -\sqrt{2}\,\Im\alpha$ and $z\nu = \sqrt{2}\,\Re\alpha$).

In order to reconstruct the usual tomographic formula for the homodyne detection,[19] we need to pass to polar variables, i.e., $\mu = -r\cos\phi$, $\nu = -r\sin\phi$; then

$$\hat{x} \to -r\hat{x}_\phi = -r[\hat{q}\cos\phi + \hat{p}\sin\phi] \tag{69}$$

Furthermore, indicating with x_ϕ the eigenvalues of the quadrature $\hat{x}_\phi$, we have

$$\operatorname{Tr}\{\hat{\rho}e^{-i\hat{x}}\} = \operatorname{Tr}\{\hat{\rho}e^{ir\hat{x}_\phi}\} = \int dx_\phi w(x_\phi, \phi)\, e^{irx_\phi} \tag{70}$$

and thus, from Eq. (66),

$$\hat{\rho} = \int d\phi\, dx_\phi w(x_\phi, \phi)\, \hat{K}_\phi \tag{71}$$

with

$$\hat{K}_\phi = \frac{1}{2\pi} \int dr\, r e^{ir(x_\phi - \hat{x}_\phi)} \tag{72}$$

which is the same as Ref. 19. In essence, the kernel of Eq. (72) is given by the radial integral of the kernel of Eq. (68), and this is due to the fact that we pass from a general transformation, with two free parameters, to a particular transformation (homodyne rotation) with only one free parameter, and then we need to integrate over the other one. This derivation follows Ref. 20.

8. GENERATING FUNCTION FOR MOMENTA

Since the marginal distribution $w(x, \mu, \nu)$ has all the properties of the classical probability density, one could calculate the highest momenta for the shifted and squeezed quadrature $\hat{x}$. We have, by definition,

$$\langle \hat{x}^n \rangle = \int x^n w(x, \mu, \nu)\, dx \tag{73}$$

thus, for the mean value ($n = 1$) we have

$$\langle \hat{x} \rangle = \int x w(x, \mu, \nu)\, dx \tag{74}$$

and for the quadrature dispersion we have

$$\sigma_x = \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2 = \int x^2 w(x, \mu, \nu)\, dx - \left[\int x w(x, \mu, \nu)\, dx \right]^2 \tag{75}$$

As in the standard probability theory,[37] to calculate the highest momenta for the shifted and squeezed quadrature one can introduces the generating function

$$G(i\lambda) = \sum_{n=0}^{\infty} \frac{(i\lambda)^n}{n!} \langle \hat{x}^n \rangle \tag{76}$$

Then the highest momenta are the coefficients of the Taylor series for the decomposition of the generating function with respect to the parameter $(i\lambda)$.

We will express this generating function in terms of the Wigner function for the quantum system. Inserting Eq. (5) into Eq. (73), we have

$$\langle \hat{x}^n \rangle = \int x^n e^{-ik(x-\mu q - \nu p)} W(q, p) \frac{dq\, dp\, dk\, dx}{(2\pi)^2} \tag{77}$$

and inserting this into Eq. (76) we arrive at

$$G(i\lambda) = \int e^{-ik(x-\mu q-\nu p)+i\lambda x} W(q, p) \frac{dq\, dp\, dk\, dx}{(2\pi)^2} \tag{78}$$

Now integrating, first over the quadrature variable x and then over the variable k, we get

$$G(i\lambda) = \int \frac{dq\, dp}{2\pi} W(q, p)\, e^{i\lambda(\mu q + \nu p)} \tag{79}$$

This expression shows that the generating function for the quadrature highest momenta is determined by the Fourier components of the system Wigner function

$$W_F(a, b) = \frac{1}{(2\pi)^2} \int W(q, p)\, e^{iqa + ipb}\, dq\, dp \tag{80}$$

i.e.,

$$G(i\lambda) = \frac{1}{2\pi} W_F(\lambda\mu, \lambda\nu) \tag{81}$$

Thus, having the Wigner function of the quantum state and calculating its Fourier component, we may determine the generating function, which depends on the extra parameters μ, ν. On the other hand, since from Eq. (80) we have the inverse Fourier transform

$$W(q, p) = \frac{1}{(2\pi)^2} \int W_F(a, b)\, e^{-iqa - ipb}\, da\, db \tag{82}$$

we can express the Wigner function trough the generating function as

$$W(q, p) = \frac{1}{(2\pi)^3} \int e^{-i\mu q - i\nu p} G(i)\, d\mu\, d\nu \tag{83}$$

where we have taken $\lambda = 1$ and integrated over the parameters μ and ν on which the generating function depends.

Hence we conclude that the quantum information about the state is completely contained in the expression for the generating function. This reflects the fact that by measuring the shifted, rotated, and squeezed quadrature, we measure the momenta of the marginal distribution $w(x, \mu, \nu)$, and in fact we can reconstruct the generating function as a function of the extra parameters μ, ν. Thus, the Wigner function of the system is obtained from Eq. (83).

9. CONDITIONAL PROBABILITY

The direct extension of classical probability concepts leads also to the notion conditional probability. Using the convention that $\mathbf{x}$ means the vector given by the quadrature variable x and the parameters μ and ν, the joint probability $w(\mathbf{x}_1, t_1; \mathbf{x}_2, t_2)$ is defined as the probability to have x_1 as a result of the quadrature measurement at time t_1 in the frame $\{\mu_1, \nu_1\}$, and x_2 as a result of the quadrature measurement at time t_2 in the frame $\{\mu_2, \nu_2\}$. Then the conditional probability follows as

$$w(\mathbf{x}_1, t_1 \,|\, \mathbf{x}_2, t_2) = \frac{w(\mathbf{x}_1, t_1; \mathbf{x}_2, t_2)}{w(\mathbf{x}_2, t_2)} \tag{84}$$

As a consequence, the Chapman–Kolmogorov equation[38] will be satisfied, i.e.,

$$w(\mathbf{x}_1, t_1) = \int d^3\mathbf{x}_2 w(\mathbf{x}_1, t_1 \,|\, \mathbf{x}_2, t_2)\, w(\mathbf{x}_2, t_2) \tag{85}$$

so that the defined conditional probability can be interpreted as the propagator for the marginal distribution. The physical meaning of the real positive propagator (84) is the following: it is the transition probability to go from the position x_2 in which the particle is situated at initial time t_2 in the reference frame labeled by scaling and rotation parameters $\{\mu_2, \nu_2\}$, to the position x_1 at the moment t_1 in the reference frame labeled by the parameters $\{\mu_1, \nu_1\}$.

We remark that, even thought the stochastic process on which the marginal distribution depends is only one, we need to integrate also on the variables representing the parameters since the same process may "come" from different frames. In fact, indeed the normalization condition, as a consequence of Eqs. (3) and (85), can be read as

$$\int dx_1\, d^3\mathbf{x}_2\, w(\mathbf{x}_1, t_1 \,|\, \mathbf{x}_2, t_2)\, w(\mathbf{x}_2, t_2) = 1 \tag{86}$$

In order to see the equation which the conditional probability (84) obeys, we insert Eq. (85) into Eq. (30), obtaining

$$w(\mathbf{x}_1, t_1 \,|\, \mathbf{x}_2, t_2)\overleftarrow{(\partial_{t_1} + \Pi(\check{p}, \check{q}))} = \delta^3(\mathbf{x}_1 - \mathbf{x}_2)\,\delta(t_1 - t_2) \tag{87}$$

that is analogous to the differential Chapman–Kolmogorov equation.[38]

As an application, let us consider the case of the driven harmonic oscillator for which, from Eq. (34), we have

$$\left(\frac{\partial}{\partial t_1} - \mu_1 \frac{\partial}{\partial v_1} + \omega^2 v_1 \frac{\partial}{\partial \mu_1} + f v_1 \frac{\partial}{\partial x_1}\right) w(\mathbf{x}_1, t_1 \,|\, \mathbf{x}_2, t_2) = \delta^3(\mathbf{x}_1 - \mathbf{x}_2)\,\delta(t_1 - t_2) \tag{88}$$

whose solution, for $t_1 > t_2$, will be

$$\begin{aligned} w(\mathbf{x}_1, t_1 \,|\, \mathbf{x}_2, t_2) = {} & \delta(v_2 - \mu_1 \sin[\omega(t_1 - t_2)] - \omega v_1 \cos[\omega(t_1 - t_2)]) \\ & \times \delta(\mu_1 \cos[\omega(t_1 - t_2)] - \omega v_1 \cos[\omega(t_1 - t_2)] - \mu_2) \\ & \times \delta\left(x_1 - x_2 - \mu_1 \frac{f}{\omega^2}\{1 - \cos[\omega(t_1 - t_2)]\}\right. \\ & \left. - v_1 \frac{f}{\omega} \sin[\omega(t_1 - t_2)]\right) \end{aligned} \tag{89}$$

Now, by means of Eqs. (85) and (89) we may derive the solution of Eq. (34) starting, for example, from an initial coherent condition characterized by q_0 and p_0, i.e., Eq. (42) at $t = 0$, obtaining

$$\begin{aligned} w(x, \mu, v) = {} & \frac{1}{\sqrt{\pi(\mu^2 + \omega^2 v^2)}} \\ & \times \exp\left\{ -\left[x - \mu \frac{f}{\omega^2}(1 - \cos(\omega t)) - v \frac{f}{\omega} \sin(\omega t) \right.\right. \\ & - q_0(\mu \cos(\omega t) - \omega v \sin(\omega t)) \\ & \left.\left. + p_0(\mu(\sin(\omega t) + \omega v \cos(\omega t)) \right]^2 \Big/ (\mu^2 + \omega^2 v^2) \right\} \end{aligned} \tag{90}$$

where we have taken $\mathbf{x}_1 = \mathbf{x}$, $t_1 = t$ and $t_2 = 0$. Of course, if we set $f = 0$ and $\omega = 1$ in Eq. (90) we have again the solution (42).

Finally, as a special case of the propagator formula (85), we can consider the time evolution of the marginal distribution of Ref. 16

$$w(x_1, \mu_1 = \cos\phi, \nu_1 = \sin\phi, t_1) = \int d^3\mathbf{x}_2\, w(x_1, \mu_1 = \cos\phi, \nu_1 = \sin\phi, t_1 \mid \mathbf{x}_2, t_2)\, w(\mathbf{x}_2, t_2) \qquad (91)$$

This can be useful as a connection between our formalism and the homodyne tomography at different times.

10. CONCLUSIONS

We have shown that it is possible to bring the quantum dynamics back to the classical description in terms of spa probability distribution containing (over)complete information. The time evolution of a measurable probability for the discussed observables can be useful both for the prediction of the experimental outcomes at a given time and, as mentioned above, to achieve the quantum state of the system at any time. Furthermore, the symplectic transformation of Eq. (1) can be represented as a composition of shift, rotation, and squeezing. So, we emphasize that our procedure allows one to transform the problem of quantum measurements (at least for some observables) into a problem of classical measurements with an ensemble of shifted, rotated, and scaled reference frames in the (classical) phase space.

Quite generally, physics distinguishes between the dynamical law and the state of a system. The state contains the complete statistical information about an ensemble of physical objects at a particular moment, while the dynamical law determines the change of the status quo at the next istant of time. But can we use the dynamical law to infer the state (for example) of a moving particle after position measurements have been performed? For instance, in molecular emission tomography,[39] the quantum state of a mulecular vibration has been determined from its elongation encoded in the time-evolved fluorescence spectrum, while the usual standard tomography schemes[17] have been restricted to harmonic oscillators or free particles for which one has a simple shearing or rotation in the phase space; however, the developed formalism is able to infer the state of a particle moving in an arbitrary potential[40] provided with position-like measurements in different frames (an analogous problem using a nontomographic approach has been studied in Ref. 41). Of course, in some situations the measurements of instantaneous values of the marginal distribution for different values of the parameters can be replaced by measuring the distribution for these parameters which change in time. Such measurements may be consistent with the system evolution if the time variation of parameters is much faster than the natural evolution of the system itself. In this case the state of the system does not change during the measurement process and one obtains the instantaneous value of the marginal distribution and that of the Wigner function.

Finally, we believe that our "classical" approach can be a powerful tool for investigating complex quantum systems as, for example, chaotic systems in which the quantum chaos can be considered in the framework of equations for a real and positive distribution function. On the other hand, since the symplectic transformations are usually involved in the theory of special relativity, we may be able to apply the developed formalism for a relativistic formulation of the quantum measurement theory. These will be the subjects of future papers.

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