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Symplectic tomography as classical approach to quantum systems

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Abstract

By using a generalization of the optical tomography technique we describe the dynamics of a quantum system in terms of equations for a purely classical probability distribution which contains complete information about the system.

1. Introduction

Due to the Heisenberg [1] and Schrödinger–Robertson [2,3] uncertainty relations for the position and momentum in quantum systems, there does not exist a joint distribution function in the phase space. Nevertheless, the wish to understand quantum mechanics in terms of classical probabilities has led to the introduction of so-called quasi-probability distributions, such as the Wigner function [4], the Husimi Q -function [5] and the Glauber–Sudarshan P -function [6,7]. Later, a set of s -ordered quasi-distributions [8] unified these quasi-probabilities into a one-parameter family. Even in the early days of quantum mechanics Madelung [9] observed that the modulus and the phase of wave functions obey the hydrodynamical classical equations, and along this line the stochastic quantization scheme has been suggested by Nelson [10] to link the classical stochastic mechanics formalism with the quantum mechanical basic entities,

such as wave function and propagator. In some sense, also hidden variables [11] were proposed to relate the quantum processes to the classical ones. Nevertheless, up till now there does not exist a formalism which consistently connects the “two worlds”.

The discussed quasi-probabilities clarified the similarities and the differences between classical and quantum considerations, and they are widely used as an instrument for calculations in quantum theory [12,13]. However, they cannot play the role of classical distributions, since, for example, the Wigner function and the P -function may have negative values. Although the Q -function is always positive and normalized, it does not describe measurable distributions of concrete physical variables.

Using the formalism of Ref. [8], Vogel and Risken [14] found an integral relation between the Wigner function and the marginal distribution for the measurable homodyne output variable which represents a rotated quadrature. This result gives the possibility of measuring the quantum state, and it is referred to as optical homodyne tomography [15].

In Ref. [16] a symplectic tomography procedure

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was suggested to obtain the Wigner function by measuring the marginal distribution for a shifted and squeezed quadrature, which depends on extra parameters. In Ref. [17] the formalism of Ref. [14] was formulated in an invariant form, relating the homodyne output distribution directly to the density operator. In Ref. [18] the symplectic tomography formalism was also formulated in this invariant form and it was extended to the multimode case. Thus, due to the introduction of a quantum tomography procedure, the real positive marginal distribution for measurable observables, such as rotated, shifted and squeezed quadratures, turned out to determine completely the quantum states.

The aim of the present work is to formulate the standard quantum dynamics in terms of the classical marginal distribution of the measurable shifted and squeezed quadrature components, used in the symplectic tomography scheme. Thus we obtain an alternative formulation of the quantum system evolution in terms of the evolution of real and positive distribution functions for measurable physical observables. We will show the connection of such a “classical” probability evolution with the evolution of the above-discussed quasi-probability distributions.

Examples relative to states of the harmonic oscillator and free motion will be considered in the frame of the given formulation of quantum mechanics.

2. Density operator and distribution for shifted and squeezed quadrature

In Ref. [16] it was shown that, for the generic linear combination of quadratures, which is a measurable observable ($\hbar = 1$)

$$\hat{X} = \mu\hat{q} + \nu\hat{p} + \delta, \quad (1)$$

where \hat{q} and \hat{p} are the position and momentum respectively, the marginal distribution $w(X, \mu, \nu, \delta)$ (normalized with respect to the X variable), depending upon three extra real parameters μ, ν, δ , is related to the state of the quantum system, expressed in terms of its Wigner function $W(q, p)$, as follows:

$$w(X, \mu, \nu, \delta) = \int e^{-ik(X - \mu q - \nu p - \delta)} W(q, p) \times \frac{dk dq dp}{(2\pi)^2}. \quad (2)$$

This formula can be inverted and the Wigner function of the state can be expressed in terms of the marginal distribution [16],

$$W(q, p) = (2\pi)^2 s^2 e^{isX} w_F(X, sq, sp, s), \quad (3)$$

where $w_F(X, a, b, s)$ is the Fourier component of the marginal distribution (2) taken with respect to the parameters μ, ν, δ , i.e.

$$w_F(X, a, b, s) = \frac{1}{(2\pi)^3} \int w(X, \mu, \nu, \delta) \times e^{-i(\mu a + \nu b + \delta s)} d\mu d\nu d\delta. \quad (4)$$

Hence, it was shown that the quantum state can be described by the positive classical marginal distribution for the squeezed, rotated and shifted quadrature. In the case of only a rotated quadrature, $\mu = \cos \phi$, $\nu = \sin \phi$ and $\delta = 0$, the usual optical tomography formula of Ref. [14] gives the same possibility through the Radon transform instead of the Fourier transform. This is, in fact, a partial case of the symplectic transformation of the quadrature since the rotation group is a subgroup of the symplectic group $ISp(2, R)$ whose parameters are used to describe the transformation (1).

In Ref. [18] an invariant form directly connecting the marginal distribution $w(X, \mu, \nu, \delta)$ and the density operator was found:

$$\hat{\rho} = \int d\mu d\nu d\delta w(X, \mu, \nu, \delta) \hat{K}_{\mu, \nu, \delta}, \quad (5)$$

where the kernel operator has the form

$$\hat{K}_{\mu, \nu, \delta} = \frac{1}{2\pi} s^2 e^{is(X - \delta)} e^{-is^2 \mu \nu / 2} e^{-is\nu \hat{p}} e^{-is\mu \hat{q}}. \quad (6)$$

Formulae (3) and (5) of the symplectic tomography show that there exists an invertible map between the quantum states described by the set of nonnegative and normalized Hermitian density operators $\hat{\rho}$ and the set of positive, normalized marginal distributions (the “classical” ones) for the measurable shifted and squeezed quadratures. So, the information contained in the marginal distribution is the same which is contained in the density operator; and due to this, one can represent the quantum dynamics in terms of the evolution of the marginal probability.

3. Quantum evolution as a classical process

We now derive the evolution equation for the marginal distribution function w using the invariant form of the connection between the marginal distribution and the density operator given by formula (5). Then from the equation of motion for the density operator

$$\partial_t \hat{\rho} = -i[\hat{H}, \hat{\rho}], \quad (7)$$

we obtain the evolution equation for the marginal distribution in the form

$$\int d\mu d\nu d\delta [\dot{w}(X, \mu, \nu, \delta, t) \hat{K}_{\mu, \nu, \delta} + w(X, \mu, \nu, \delta, t) \hat{I}_{\mu, \nu, \delta}] = 0, \quad (8)$$

in which the known Hamiltonian determines the kernel $\hat{I}_{\mu, \nu, \delta}$ through the commutator

$$\hat{I}_{\mu, \nu, \delta} = i[\hat{H}, \hat{K}_{\mu, \nu, \delta}]. \quad (9)$$

The obtained integral-operator equation for simple cases can be reduced to the partial differential equation. To do this we represent the kernel operator $\hat{I}_{\mu, \nu, \delta}$ in a normal order form (i.e. all the momentum operators on the left side and the position ones on the right side) containing the operator $\hat{K}_{\mu, \nu, \delta}$ as follows:

$$: \hat{I}_{\mu, \nu, \delta} := \mathcal{R}(\hat{p}) : \hat{K}_{\mu, \nu, \delta} : \mathcal{P}(\hat{q}), \quad (10)$$

where $\mathcal{R}(\hat{p})$ and $\mathcal{P}(\hat{q})$ are finite or infinite operator polynomials (depending also on the parameters μ and ν) determined by the Hamiltonian. Then, calculating the matrix elements of the operator equation (8) between the states $\langle p|$ and $|q\rangle$ and using the completeness property of the Fourier exponents, we arrive at the following partial differential equation for the marginal distribution function:

$$\partial_t w + \Pi(\tilde{p}, \tilde{q}) w = 0, \quad (11)$$

where the polynomial $\Pi(\tilde{p}, \tilde{q})$ is the product of the polynomials $\mathcal{R}(p)$ and $\mathcal{P}(q)$ represented in the form

$$\Pi(p, q) = \mathcal{R}(p) \mathcal{P}(q) = \sum_n \sum_m p^n q^m c_{n,m}(\mu, \nu), \quad (12)$$

in which the c -number variables p and q should be replaced by the operators

$$\begin{aligned} \tilde{p} &= \left(\frac{1}{\partial/\partial\delta} \frac{\partial}{\partial\nu} + i \frac{\mu}{2} \frac{\partial}{\partial\delta} \right), \\ \tilde{q} &= \left(\frac{1}{\partial/\partial\delta} \frac{\partial}{\partial\mu} + i \frac{\nu}{2} \frac{\partial}{\partial\delta} \right), \end{aligned} \quad (13)$$

where the derivative in the denominator is understood as an integral operator. One should point out that the operators \tilde{p} and \tilde{q} in Eq. (11) act on the product of coefficients $c_{n,m}(\mu, \nu)$ and the marginal distribution corresponding to the order shown by Eqs. (11) and (12). Let us consider the important example of the particle motion in a potential with the Hamiltonian

$$\hat{H} = \frac{1}{2} \hat{p}^2 + V(\hat{q}); \quad (14)$$

then the described procedure of calculating the normal order kernel (10) gives the following form of the quantum dynamics in terms of a Fokker–Planck-like equation for the marginal distribution:

$$\begin{aligned} \dot{w} - \mu \frac{\partial}{\partial\nu} w - i \left[V \left(\frac{1}{\partial/\partial\delta} \frac{\partial}{\partial\mu} + i \frac{\nu}{2} \frac{\partial}{\partial\delta} \right) \right. \\ \left. - V \left(\frac{1}{\partial/\partial\delta} \frac{\partial}{\partial\mu} - i \frac{\nu}{2} \frac{\partial}{\partial\delta} \right) \right] w = 0, \end{aligned} \quad (15)$$

which in general is an integro–differential equation. For the free motion, $V = 0$, this evolution equation becomes the first-order partial differential equation

$$\dot{w} - \mu \frac{\partial}{\partial\nu} w = 0. \quad (16)$$

For the harmonic oscillator $V(\hat{q}) = \hat{q}^2/2$, the quantum dynamic equation has the form

$$\dot{w} - \mu \frac{\partial}{\partial\nu} w + \nu \frac{\partial}{\partial\mu} w = 0. \quad (17)$$

Thus given a Hamiltonian of the form (14) we can study the quantum evolution of the system writing down a Fokker–Planck-like equation for the marginal distribution. Solving this for a given initial positive and normalized marginal distribution, we can obtain the quantum density operator $\hat{\rho}(t)$ according to Eq. (5). Conceptually it means that we can discuss the system quantum evolution considering classical real positive and normalized distributions for the measurable variable X which is a shifted and squeezed quadrature. The distribution function which depends on extra parameters obeys a classical equation which preserves the normalization condition of the distribution. In this

sense we always can reduce the quantum behaviour of the system to the classical behaviour of the marginal distribution of the shifted and squeezed quadrature. Of course, this statement respects the uncertainty relation, because the measurable marginal distribution is the distribution for one observable. That is the essential difference (despite some similarity) of the introduced marginal distribution from the discussed quasi-distributions, including the real positive Q -function, which depend on the two variables of the phase space and are normalized with respect to these variables. We would point out that we do not derive quantum mechanics from classical stochastic mechanics, i.e. we do not quantize any classical stochastic process, our result is to present the quantum dynamics equations as classical ones, and in doing this we need not only a classical Hamiltonian but also its quantum counterpart.

4. Examples

Below we consider simple examples of the marginal distribution evolution for states of free motion and the harmonic oscillator. First of all we take into account the free motion for which Eq. (16) has a Gaussian solution of the form

$$w(X, \mu, \nu, \delta, t) = \frac{1}{\sqrt{2\pi\sigma_X(t)}} \exp\left(-\frac{(X-\delta)^2}{2\sigma_X(t)}\right), \quad (18)$$

where the dispersion of the observable X depends on time and parameters as follows:

$$\sigma_X(t) = \frac{1}{2} [\mu^2 (1+t^2) + \nu^2 + 2\mu\nu t]. \quad (19)$$

The initial condition corresponds to the marginal distribution of the ground state of an artificial harmonic oscillator calculated from the respective Wigner function [16].

If we consider the first excited state of the harmonic oscillator, we know the Wigner function [19]

$$W_1(q, p) = -2(1 - 2q^2 - 2p^2) \exp(-q^2 - p^2). \quad (20)$$

This result is time independent due to the stationarity of the state, but for small q and p it becomes negative, while the solution of Eq. (17),

$$w_1(X, \mu, \nu, \delta, t) = \frac{2}{\sqrt{\pi}} (\mu^2 + \nu^2)^{-3/2} (X - \delta)^2 \times \exp\left(-\frac{(X - \delta)^2}{\mu^2 + \nu^2}\right), \quad (21)$$

is itself time independent, but everywhere positive.

Indeed, a time evolution is present explicitly in the coherent state, whose Wigner function is given by

$$W_c(q, p) = 2 \exp[-q^2 - q_0^2 - p^2 - p_0^2 + 2(qq_0 + pp_0) \cos t - (pq_0 - qp_0) \sin t], \quad (22)$$

where q_0 and p_0 are the initial values of position and momentum. For the same state, the marginal distribution shows a more complicated evolution,

$$w_c(X, \mu, \nu, \delta, t) = \frac{1}{\sqrt{\pi}} (\mu^2 + \nu^2)^{-1/2} \times \exp\left(-q_0^2 - p_0^2 - \frac{(X - \delta)^2}{\nu^2} + 2\frac{X - \delta}{\nu} (p_0 \cos t - q_0 \sin t)\right) \times \exp\left[\frac{1}{\mu^2 + \nu^2} \left(\frac{\mu}{\nu} (X - \delta) + q_0(\mu \sin t + \nu \cos t) + p_0(\nu \sin t - \mu \cos t)\right)^2\right]. \quad (23)$$

It is also interesting to consider the comparison between Wigner function and marginal probability for non-classical states of the harmonic oscillator, such as the female cat state defined as [20]

$$|\alpha_-) = N_- (|\alpha) - |-\alpha), \quad \alpha = 2^{-1/2}(q_0 + ip_0), \quad (24)$$

with

$$N_- = \left(\frac{\exp[(q_0^2 + p_0^2)/2]}{4 \sinh[(q_0^2 + p_0^2)/2]}\right)^{1/2}, \quad (25)$$

and for which the Wigner function assumes the following form:

$$\begin{aligned}
W_-(q, p) &= 2N_-^2 e^{-q^2 - p^2} \\
&\times \left(e^{-q_0^2 - p_0^2} \cosh[2(qq_0 + pp_0) \cos t \right. \\
&+ 2(qp_0 - pq_0) \sin t] - \cos[2(qp_0 - pq_0) \cos t \\
&\left. - 2(qq_0 + pp_0) \sin t] \right). \quad (26)
\end{aligned}$$

The corresponding marginal distribution is

$$\begin{aligned}
w_-(X, \mu, \nu, \delta, t) &= N_-^2 [w_A(X, \mu, \nu, \delta, t) \\
&- w_B(X, \mu, \nu, \delta, t) - w_B^*(X, \mu, \nu, \delta, t) \\
&+ w_A(-X, \mu, \nu, -\delta, t)], \quad (27)
\end{aligned}$$

with

$$\begin{aligned}
w_A(X, \mu, \nu, \delta, t) &= \frac{1}{\sqrt{\pi}} (\mu^2 + \nu^2)^{-1/2} \\
&\times \exp \left(-q_0^2 - p_0^2 - \frac{(X - \delta)^2}{\nu^2} \right. \\
&+ 2 \frac{X - \delta}{\nu} (p_0 \cos t - q_0 \sin t) \left. \right) \\
&\times \exp \left[\frac{1}{\mu^2 + \nu^2} \left(\frac{\mu}{\nu} (X - \delta) \right. \right. \\
&+ q_0 (\mu \sin t + \nu \cos t) \\
&\left. \left. + p_0 (\nu \sin t - \mu \cos t) \right)^2 \right], \quad (28)
\end{aligned}$$

and

$$\begin{aligned}
w_B(X, \mu, \nu, \delta, t) &= \frac{1}{\sqrt{\pi}} (\mu^2 + \nu^2)^{-1/2} \\
&\times \exp \left(-\frac{(X - \delta)^2}{\nu^2} \right. \\
&- 2i \frac{X - \delta}{\nu} (q_0 \cos t + p_0 \sin t) \left. \right) \\
&\times \exp \left[-\frac{1}{\mu^2 + \nu^2} \left(-i \frac{\mu}{\nu} (X - \delta) \right. \right. \\
&+ q_0 (\mu \cos t - \nu \sin t) \\
&\left. \left. + p_0 (\mu \sin t + \nu \cos t) \right)^2 \right]. \quad (29)
\end{aligned}$$

The presented examples show that with the evolution of the state of a quantum system, one can always associate the evolution of the probability density for the random classical variable X which obeys a “classical” Fokker–Planck-like equation, and this probability density contains the same information (about the quantum

system) which is contained in any quasi-distribution function. But the probability density has the advantage of behaving completely as the usual classical one. The physical meaning of the “classical” random variable X is transparent, it is considered as the position in an ensemble of shifted, rotated and scaled rest frames in the classical phase space of the system under study. We remark that for non-normalized quantum states, like the states with fixed momentum (De Broglie wave) or with fixed position, the introduced map in Eq. (5) may be preserved. In this context the plane wave states of free motion have the marginal distribution corresponding to classical white noise.

5. Conclusions

We have shown that it is possible to bring the quantum dynamics back to a classical description in terms of a probability distribution containing (over)complete information. The time evolution of a measurable probability for the discussed observables can be useful both for the prediction of the experimental outcomes at a given time and, as mentioned above, to achieve the quantum state of the system at any time. Furthermore the symplectic transformation of Eq. (1) can be represented as a composition of shift, rotation and squeezing. So, the measurement of a shifted variable means the measure of the coordinate in a frame in which the zero is shifted. This could be implemented for example by measuring the oscillator coordinate using an infinite ensemble of frames which are shifted with respect to the initial one (a related method was discussed also in Ref. [21]). Furthermore if one considers the variable \hat{q} as the photon quadrature, which corresponds to the amplitude of the electric vector vibrations, a rotation means a homodyne measurement, while the squeezing means measurement after amplification or attenuation. So, we want to emphasize that our procedure allows us to transform the problem of quantum measurements (at least for some observables) into the problem of classical measurements with an ensemble of shifted, rotated and scaled reference frames in the (classical) phase space.

We also want to remark that in some situations the measurements of instantaneous values of the marginal distribution for different values of the parameters is re-

placed by measuring the distribution for these parameters which evolve in time. Such measurements may be consistent with the system evolution if the parameter's time variation is much faster than the natural evolution of the system itself. In this case the state of the system does not change during the measurement process and one obtains the instant value of the marginal distribution and of the Wigner function.

Finally we believe that our "classical" approach could be a powerful tool to investigate complex quantum system as for example chaotic systems in which the quantum chaos could be considered in a frame of equations for a real and positive distribution function.

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