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Wigner function and probability distribution for shifted and squeezed quadratures

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Abstract. The probability distribution for rotated, squeezed and shifted quadratures is shown to be expressed in terms of the Wigner function (as well as in terms of the Q-function and density operator in the coordinate representation). The inverse transformation generalizing the homodyne detection formula is obtained.

1. Introduction

The homodyne measurement of an electromagnetic field gives all possible linear combinations of the field quadratures just by varying the phase of the local oscillator. The average of the random outcomes of the measurement, at a given local oscillator phase, is connected with the marginal distribution of the Wigner function, or any other quasi-probability used in quantum optics. In the work [1] it was shown, indeed, that the rotated quadrature phase may be expressed in terms of the Wigner function [2] as well as in terms of the Husimi Q-function [3] and Glauber–Sudarshan P-representation [4, 5]. The result of [1] was based on the observation that relations of the density matrix in different representations to the characteristic function, which is the mean value of the displacement operator creating a coherent state from the vacuum [4] (studied in [6]), may be rewritten as relations of the characteristic function, which is the mean value of the rotated quadrature phase. It gave the possibility of expressing the Wigner function in terms of the marginal distribution of homodyne outcomes through the tomographic formula. The essential point of the obtained formula is that the homodyne output distribution may be measured, and the corresponding inverse Radon transform produces the Wigner distribution function which is referred to as the optical homodyne tomography [7]. The density matrix elements, in some representation, can also be obtained by avoiding the Wigner function and the Radon transform [8]. On the other hand, the structure of the relations of the characteristic functions to the density operators, in different representations, shows that the relation is based on a linear transformation of both quadratures which is a simple rotation determined by an angle φ (we discuss one-mode systems).

The aim of this work is to extend the results of [1] to the case of a generic linear transformation of quadratures belonging to the Lie group ISp(2, R), and to introduce the extra parameters, which include squeezing and shifting of quadratures, into the formulae relating analogues of homodyne output with the Wigner function, Q-function and density matrix of the system under study. This will permit more accessible ways of measuring the radiation field and then obtaining a larger amount of

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information on the quasi-probability distribution one wishes to reconstruct. To have various measurement procedures may be convenient in some experiments where not only the radiation field but also a mechanical oscillator should be measured.

2. Density matrix and characteristic function

We consider in this section the following problem. Given the density matrix $\rho(x,x')$ of a quantum system, find the characteristic function which is by definition the mean value of the exponent of the Hermitian linear form $\hat{X}$ in position $\hat{q}$ and momentum $\hat{p}$

$$F(k)=\langle \mathrm{e}^{\mathrm{i}k\hat{X}}\rangle=\mathrm{Tr}\,\rho\,\mathrm{e}^{\mathrm{i}k\hat{X}} \tag{1}$$

where the observable

$$\hat{X}=\mu\hat{q}+\nu\hat{p}+\delta \tag{2}$$

is considered to be measured by some experimental device. Then, the Fourier transform of the function $F(k)$

$$w(X,\mu,\nu,\delta)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(k)\,\mathrm{e}^{-\mathrm{i}kX}\,\mathrm{d}k \tag{3}$$

is the marginal distribution function for the variable X (the eigenvalue of the observable $\hat{X}$) depending on parameters of the linear transform.

The direct calculation gives for the characteristic function the expression

$$F(k)=\mathrm{e}^{(\mathrm{i}k^2\mu\nu)/2+\mathrm{i}\delta k}\int_{-\infty}^{\infty}\rho(x+\nu k,x)\,\mathrm{e}^{\mathrm{i}kx\mu}\,\mathrm{d}x. \tag{4}$$

The marginal distribution function (3) for the observable (2) is related to the density matrix in the coordinate representation by the expression

$$w(X,\mu,\nu,\delta)=\frac{1}{2\pi}\int\int_{-\infty}^{\infty}\exp\left(-\mathrm{i}kX+\frac{1}{2}\mu\nu k^2+\mathrm{i}ky\mu+\mathrm{i}k\delta\right)\rho(y+\nu k,y)\,\mathrm{d}y\,\mathrm{d}k. \tag{5}$$

For the conjugate variable

$$\hat{\mathcal{P}}=\mu'\hat{q}+\nu'\hat{p}+\delta'$$

the marginal distribution is expressed in terms of the density matrix as

$$w(\mathcal{P},\mu',\nu',\delta')=\frac{1}{2\pi}\int\int_{-\infty}^{\infty}\exp\left(-\mathrm{i}k\mathcal{P}+\frac{1}{2}\mu'\nu'k^2+\mathrm{i}ky\mu'+\mathrm{i}k\delta'\right)\times\rho(y+\nu'k,y)\,\mathrm{d}y\,\mathrm{d}k. \tag{6}$$

The parameters δ, δ' are real shift parameters, while for the others we can introduce the matrix

$$\Lambda=\begin{pmatrix}\mu & \nu\\ \mu' & \nu'\end{pmatrix}$$

which is a real symplectic matrix, i.e.

$$\Lambda\sigma\Lambda^{\mathrm{T}}=\sigma$$

where

$$\sigma=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

3. Wigner function and marginal distribution

In this section we will consider the expression for the marginal distribution of the variable $\hat{X}$, equation (2), in terms of the given Wigner function which is related to the density matrix in the coordinate representation by the formula ($\hbar=1$)

$$W(q,p)=\int_{-\infty}^{\infty} \rho(q+\tfrac{1}{2}u, q-\tfrac{1}{2}u)\, e^{-ipu}\, du. \tag{7}$$

The inverse transform is

$$\rho(q,q')=\frac{1}{2\pi}\int_{-\infty}^{\infty} W\left(\frac{q+q'}{2},p\right) e^{ipu}\, dp. \tag{8}$$

The Wigner function is normalized

$$\int W(q,p)\frac{dq\, dp}{2\pi}=1. \tag{9}$$

The considered characteristic function, equation (1), is expressed in terms of the Wigner function by the relation [6]

$$\langle e^{ik\hat{X}}\rangle=\int e^{ik(\mu q+\nu p+\delta)} W(q,p)\frac{dq\, dp}{2\pi}. \tag{10}$$

Then the marginal distribution of the variable X, which depends on parameters of the linear transform, is given by the relation

$$w(X,\mu,\nu,\delta)=\int e^{-ik(X-\mu q-\nu p-\delta)} W(q,p)\frac{dk\, dq\, dp}{(2\pi)^2}. \tag{11}$$

This integral is another expression for the marginal distribution given in terms of the density matrix, equation (5). The distribution is normalized as

$$\int_{-\infty}^{\infty} w(X,\mu,\mu,\delta)\, dX=1. \tag{12}$$

The observable $\hat{X}$ is considered to be measurable and the ensemble of measurements taken for all the real parameters μ,ν,δ produces experimentally the marginal distributions $w(X,\mu,\nu,\delta)$. In [1] the parameters μ, ν, δ have been chosen as

$$\mu=\cos\varphi \qquad \nu=-\sin\varphi \qquad \delta=0.$$

Then, the inverse formula determining the Wigner function in terms of the marginal distribution was obtained as the tomographic integral. In this section we will obtain the Wigner function in terms of the marginal distribution of the other observable $\hat{X}$, equation (2), containing two extra parameters which may be treated as the influence of the squeezing parameter r, and the shift δ on the quadrature, if one takes

$$\mu = e^{-r} \cos\varphi \tag{13}$$

$$\nu = e^{r} \sin\varphi. \tag{14}$$

In the absence of squeezing, i.e. $r=0$ and shift $\delta=0$, we have the homodyne detecting variables. The parameters μ, ν and δ vary from $-\infty$ to ∞.

Our aim is to express the Wigner function $W(q,p)$ if we know, from given measurements, the marginal distribution function $w(X,\mu,\nu,\delta)$ in the domain of linear transform parameters. This relation will be obtained if we make the following steps. First we consider the Fourier transform of the marginal distribution as

$$w(X,m,n,s) = \frac{1}{(2\pi)^3} \int w(X,\mu,\nu,\delta)\, e^{-im\mu - in\nu - is\delta}\, d\mu\, d\nu\, d\delta \tag{15}$$

then using equation (11) and the formula

$$\delta(m-sq)\delta(n-sp) = \frac{1}{s^2}\delta(q-m/s)\delta(p-n/s) \tag{16}$$

we arrive at

$$W(m/s, n/s) = (2\pi)^2 s^2\, e^{isX} w(X,m,n,s). \tag{17}$$

Finally, defining the variables

$$Q = m/s \qquad \mathcal{P} = n/s \qquad s=1 \qquad X=0 \tag{18}$$

we have the usual Wigner function as

$$W(Q,\mathcal{P}) = (2\pi)^2 w(0,Q,\mathcal{P},1). \tag{19}$$

Because this one is expressed as a particular Fourier component of the marginal distribution, it means that this contains overcomplete information about the system under study.

Let us now consider as an example the ground state of the oscillator with wavefunction

$$\psi_0(x) = \frac{1}{\pi^{1/4}} e^{-x^2/2}. \tag{20}$$

The Wigner function can easily be calculated

$$W(q,p) = \int \rho(q+\tfrac{1}{2}u, q-\tfrac{1}{2}u)\, e^{-ipu}\, du = 2e^{-(q^2+p^2)}. \tag{21}$$

Also the characteristic function can be obtained in this state

$$\langle e^{ik\hat{X}} \rangle = e^{ik\delta} \exp[-\tfrac{1}{4}k^2(\mu^2+\nu^2)] \tag{22}$$

then, the marginal distribution, by equation (3), will be

$$w(X,\mu,\nu,\delta) = [\pi(\mu^2+\nu^2)]^{-1/2} \exp\left(-\frac{(X-\delta)^2}{\mu^2+\nu^2}\right). \tag{23}$$

After calculating the Fourier components in δ, μ, ν variables we have

$$w(X,m,n,s)=\frac{1}{2\pi^2}\frac{1}{s^2}e^{-isX}\exp(-(m/s)^2-(n/s)^2) \tag{24}$$

which obviously gives our initial Wigner function of equation (21) if we make the same choice of equation (18).

4. Observables

The matrix Λ may be represented as the product of three matrices

$$\Lambda=\begin{pmatrix}\cos\psi & -\sin\psi\\ \sin\psi & \cos\psi\end{pmatrix}\begin{pmatrix}e^{-r} & 0\\ 0 & e^{r}\end{pmatrix}\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}.$$

Thus the observables $\hat{X}$ and $\hat{\mathcal{P}}$, of the form

$$\begin{pmatrix}\hat{X}\\ \hat{\mathcal{P}}\end{pmatrix}=\Lambda\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix}+\begin{pmatrix}\delta\\ \delta'\end{pmatrix}$$

physically correspond to the following measurements. First we consider a shift δ for the quadrature $\hat{q}$. It means the possibility of measuring the observable

$$\hat{q}_\delta=\hat{q}+\delta$$

where $\hat{q}$ is the oscillator coordinate and δ is the shift. The measurement of the variable q_δ is the measure of the coordinate in a frame in which zero is shifted. Then, to make measurements of the observable $\hat{q}_\delta$ implies measuring the oscillator coordinate using an infinite ensemble of frames which are shifted with respect to the initial one.

To measure the observable

$$\hat{p}_{\delta'}=\hat{p}+\delta'$$

implies measuring the oscillator momentum in reference frames moving with constant velocities. If one considers the photon quadrature $\hat{q}$, which corresponds to the amplitude of the electric vector vibrations, in an electromagnetic wave, by measuring the variable q_δ it implies making a measurement of the field strength using shifted values of the frame field.

The matrix

$$\Lambda_\varphi=\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}$$

produces the homodyne observable $\hat{X}_\varphi$, the measurement of which is discussed in [7]. Thus, to understand the meaning of measuring the observable $\hat{X}$ we need to understand what is the procedure to measure

$$\hat{q}_r=e^r\hat{q}$$

where r is the squeezing parameter.

When the squeezing parameter r is negative it represents the action of an amplifier, and if it is positive it is the action of an attenuator. Now, the measurement of the observable $\hat{X}$ containing three parameters μ, ν, δ means the following measurements. First, a homodyne detection where the phase angle φ is produced. Then, the new quadrature is measured after an amplification with gain e^{-r}. Finally, the obtained quadrature is again subject to another homodyne detection determined by the phase angle ψ. After this, the obtained quadrature is measured in the shifted reference frame with the parameter δ. The produced measurement yields overcomplete information about the state of the system which, nevertheless, must be self-consistent in determining the Wigner function. One could select some set of parameters to have enough measurements to reconstruct the density matrix. An example of homodyne detection is just such a subset. But, in principle, there are other subsets of overcomplete measurements which may be convenient in some situations.

It is also possible to have another set of devices transforming the signal (observables $\hat{q}$ and $\hat{p}$), and that is based on the Λ-matrix decomposition

$$\Lambda=\begin{pmatrix}1 & 0\\ a & 1\end{pmatrix}\begin{pmatrix}e^{-r} & 0\\ 0 & e^{r}\end{pmatrix}\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}$$

which holds for any matrix belonging to the group $Sp(2, R)$. Here, instead of the combination of two homodyne detections and amplification (attenuation), one could use the combination of signal transformation by a device which measures the oscillator coordinate in a moving frame. Its velocity is determined by the parameter b of the first quadrature; then by the action of an amplifier followed by the same device (with velocity $1/a$). The use of different measurement procedures may be convenient in experiments with either electromagnetic field oscillators or mechanical oscillators.

5. Comparison with the homodyne detection formula of [1]

In this section we will consider the same example of the quantum state as was studied in [1] by the tomographic formula. The superposition state of the form

$$|\Psi\rangle=N^{-1/2}(|\alpha\rangle+|\beta\rangle) \tag{25}$$

$$N=2+2\exp\left(-\tfrac{1}{2}|\alpha-\beta|^2\right)\cos(\mathrm{Im}\,\alpha^*\beta) \tag{26}$$

where $|\alpha\rangle$ and $|\beta\rangle$ are the coherent state vectors has been discussed in [6] where the Wigner function (P-distribution and Q-function) of this state has been obtained. The partial cases of these states for $\beta=-\alpha$ are even coherent states [9], or Schrödinger cat states [10, 11]

$$|\alpha_+\rangle=\{2[1+\exp(-2|\alpha|^2)]\}^{-1/2}(|\alpha\rangle+|-\alpha\rangle) \tag{27}$$

and for $\beta=\alpha^*$ the states become the same as discussed in [1]

$$|\Psi\rangle=\frac{|a+\mathrm{i}b\rangle+|a-\mathrm{i}b\rangle}{\{2[1+\cos(2ab)\exp(-2b^2)]\}^{1/2}}. \tag{28}$$

The Wigner function of this state is [1, 6]

$$W(q,p)=\frac{\exp[-2(q-a)^2]}{\pi[1+\cos(2ab)\exp(-2b^2)]}[\exp(-2(p-b)^2)+\exp(-2(p+b)^2) +2\exp(-2p^2)\cos(4bq-2ab)]. \tag{29}$$

We now calculate the marginal distribution for the state in equation (28). It is given by the formula

$$w(x,\mu,\nu,\delta)=(2/\pi)^{1/2}\left[\frac{1}{\mu^2+\nu^2}\right]^{1/2}\frac{1}{[1+\cos(2ab)\exp(-2b^2)]}$$
$$\times\exp\left(-2\frac{(x-\delta-\mu a)^2+b^2\nu^2}{\mu^2+\nu^2}\right)\times\left[\cosh\left(\frac{4\nu b(x-\delta-\mu a)}{\mu^2+\nu^2}\right)\right.$$
$$\left.+\cos\left(\frac{2b[2\mu x-2\mu\delta-a(\mu^2-\nu^2)]}{\mu^2+\nu^2}\right)\right]. \tag{30}$$

When $\mu=\cos\varphi$, $\nu=\sin\varphi$, $\delta=0$, we reproduce the result of equation (16) of [1].

The plots of the marginal distribution of equation (30) are shown for different values of μ and ν at $\delta=0$ in figures 1 and 2. Taking into account $\delta\neq 0$ it produces the shift of all plots by this amount. In figure 1 we have exactly reproduced some graphs of [1] for the distribution w. In figure 2 we show the graphs of the same function, but for values of parameters outside the range $\{-1,1\}$. The distribution shows rapid oscillations when μ is quite different from ν, whereas when ν approaches μ two maxima appear. It should be noted that for ν greater than μ we have no oscillations due to the fact that we have considered only the quadrature $\hat{X}$ (and not $\hat{\mathcal{P}}$). As another example we will give the marginal distribution function of the variable X in equation (2) for the

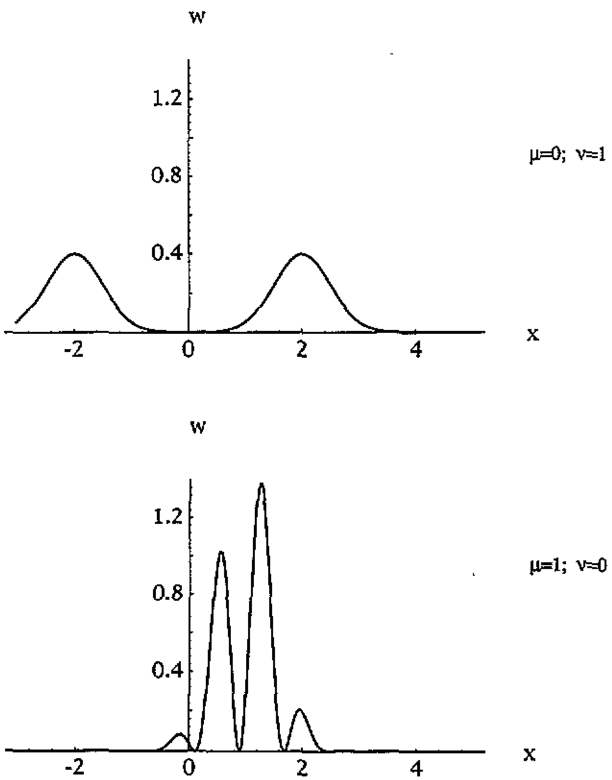

Figure 1. The distribution w is plotted against X for the indicated values of the parameters μ, ν and for $a=1$, $b=2$. This coincides with the distribution of [1].

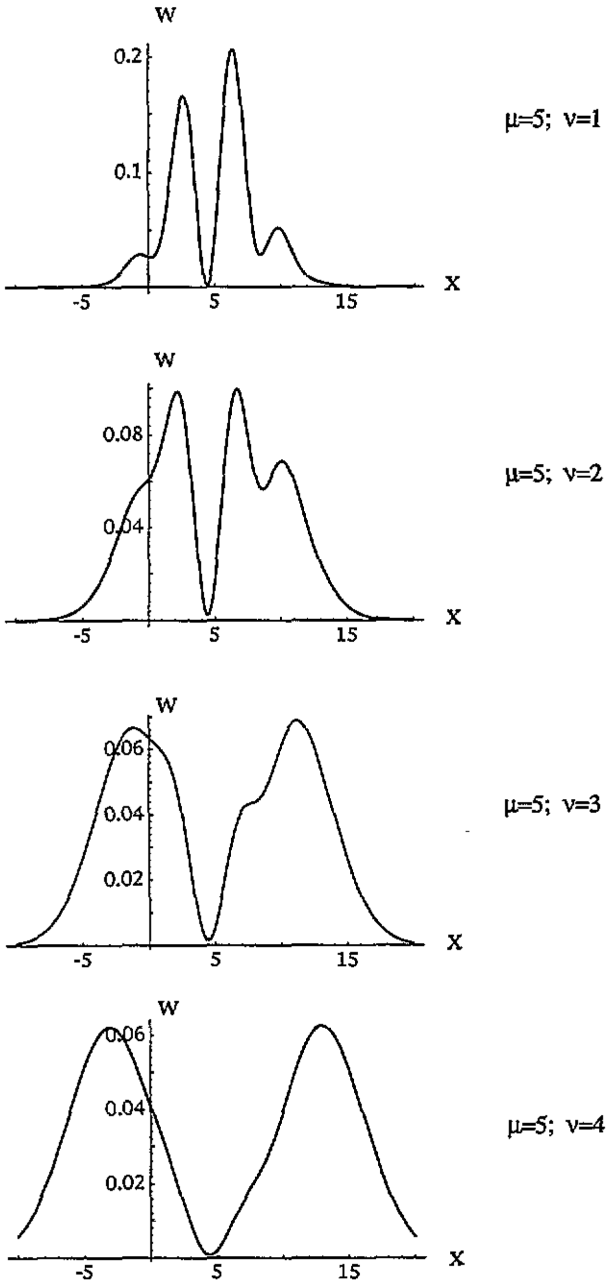

Figure 2. The distribution w is plotted against X for the indicated values of the parameters μ, ν and is different from the distribution of [1].

even coherent state [9]. Since, due to [6], the Wigner function for this state is

$$W_{+}(q,p)=4N_{+}^{2}\,\mathrm{e}^{-p^{2}-q^{2}}[\mathrm{e}^{-2|\alpha|^{2}}\cosh\{2^{3/2}\,\mathrm{Re}[\alpha(q-\mathrm{i}p)]\}+\cosh\{2^{3/2}\,\mathrm{Im}[\alpha(q-\mathrm{i}p)]\}]$$

$$N_{+}=\frac{\exp\left(\frac{1}{2}|\alpha|^{2}\right)}{2(\cosh|\alpha|^{2})^{1/2}} \qquad (31)$$

we obtain for the marginal distribution the result

$$w(X,\mu,\nu,\delta)=\frac{2}{|\mu|}N_+^2\left(\frac{\pi}{1+B^2}\right)^{1/2}\mathrm{e}^{-A^2}[w_1(\alpha)+w_1(-\alpha)+w_2(\alpha)+w_2^*(\alpha)]$$
$$A=\frac{X-\delta}{\mu}\qquad B=\frac{\nu}{\mu} \tag{32}$$

where

$$w_1(\alpha)=\exp\left(-2|\alpha|^2+2\sqrt{2}\,\mathrm{Re}\,\alpha A+\frac{b_1^2}{4A}\right)$$
$$b_1=2AB-2\sqrt{2}\,\mathrm{Re}[a(B+\mathrm{i})] \tag{33}$$

and

$$w_2(\alpha)=\exp\left(\mathrm{i}2\sqrt{2}\,\mathrm{Im}(A\alpha)+\frac{b_2^2}{4A}\right)\qquad b_2=2AB-2\mathrm{i}\sqrt{2}\,\mathrm{Im}[\alpha(B+\mathrm{i})]. \tag{34}$$

6. Conclusion

As we have shown, there is the possibility of reconstructing the state of a quantum system by measuring marginal distributions, for different sets of parameters, for observables which are generic linear combinations of quadratures. The Wigner function (and consequently the density matrix and Q-function) of the system is completely determined by the ensemble of the measured marginal distributions. Since the suggested procedure contains an overcomplete set of the measured parameters, it could give the possibility of avoiding the difficulties of the state reconstruction [12] due to reduced errors in the appropriate subsets of parameters.

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