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## Dynamics of quantum tomography in an open system

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# Dynamics of quantum tomography in an open system 

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#### Abstract

In this study, we provide a way to describe the dynamics of quantum tomography in an open system with a generalized master equation, considering a case where the relevant system under tomographic measurement is influenced by the environment. We apply this to spin tomography because such situations typically occur in $\mu \mathrm{SR}$ (muon spin rotation/relaxation/resonance) experiments where microscopic features of the material are investigated by injecting muons as probes. As a typical example to describe the interaction between muons and a sample material, we use a spin-boson model where the relevant spin interacts with a bosonic environment. We describe the dynamics of a spin tomogram using a time-convolutionless type of generalized master equation that enables us to describe short time scales and/or low-temperature regions. Through numerical evaluation for the case of Ohmic spectral density with an exponential cutoff, a clear interdependency is found between the time evolution of elements of the density operator and a spin tomogram. The formulation in this paper may provide important fundamental information for the analysis of results from, for example, $\mu \mathrm{SR}$ experiments on short time scales and/or in low-temperature regions using spin tomography.


Keywords: tomography, muons, quantum dissipative systems

## 1. Introduction

The relationship between quantum states and classical probability functions has been intensively studied since the 1930s. Two important early approaches were those of Pauli [1] and Wigner [2]. Pauli discussed the possibility of determining a wave function for a quantum particle using probability functions of position $W(\vec{x})$ and momentum $W(\vec{p})$ [1]. Because these functions do not describe the uncertainty principle between the position and momentum, Pauli's approach had limited success. Wigner attempted to identify a probability function with a generating function for a spatial correlation function. However, because this function can take negative values, it is categorized as a quasi-probability function. On the basis of these early approaches, there have subsequently been numerous quasi-probability functions proposed, such as the Hushimi Q-function [3] and the Sudarshan-Glauber P-function [4, 5].

The problem of the Wigner function being able to take negative values was solved using a Radon transform, which
enables a positive probability function to be obtained by integrating along straight lines in phase space and reconstructing the Wigner function with the slices [6]. Such a tomographic concept was also proposed to reconstruct a Wigner function, Hushimi Q-function and Sudarshan-Glauber P-function using a probability distribution for a rotated quadrature phase [7], which has been demonstrated to reconstruct a vacuum and/or squeezed light state with Homodyne detection [8, 9].

Although it has been demonstrated that a quantum state can be reconstructed with a measurable probability distribution (called a tomogram), the subject system was limited to a light field. Such a limitation can be removed by the generalization of the rotation of quadratures to a symplectic transform to include shift and squeezing [10]. This generalization led to various developments such as the dynamics of tomography [11, 12], spin systems [13-17] as well as tomography on curved surfaces $[18,19]$. The association between the density operator and the spin tomogram can be considered as a special case of mapping from Hilbert space operators
onto a space of ordinary functions equipped with a star product, which provides us with a way to map the time evolution of density operators with an ordinary function [20-22]. The spin tomogram is directly related to measurements of the $\mu \mathrm{SR}$ (muon spin rotation/relaxation/resonance) method [23]. Tomography has been extended to incorporate reduced dynamics under environmental influences with dynamical maps $[22,24]$ and a relaxation process using a Markovian approximation [25, 26].

With the development of experimental techniques, the observable times and temperatures have become shorter and lower, respectively. For example, the muon spin relaxation for a spin $\frac{1}{2}$ Kagomé lattice has been experimentally studied at low temperatures from $T \sim 50 \mathrm{mK}$ to a few K to describe the non-Markovian time evolution, which cannot be described by exponential decay [27]. To describe such a relaxation process, it is necessary to extend the tomographic representation beyond the Markovian approximation.

In this work, we provide an approach to extend the tomographic representation to an open system to treat the non-Markovian relaxation process using a generalized master equation. We apply the obtained formalism to a spin-boson model [28, 29], which has been extensively used in condensed matter physics. We describe the dynamics of spin tomograms to explain the graphical representation of the probability function for a spin system, which can provide valuable fundamental information for muon relaxation experiments.

This paper is organized as follows: in section 2, we provide an approach to quantum tomography for an open quantum system. We apply this to a spin tomogram in section 3. In section 4, we present a description of the time evolution of the spin tomogram for a spin-boson system. This is followed by a numerical evaluation in section 5 . Concluding remarks are presented in section 6 .

## 2. An approach to quantum tomography for open quantum system

In this section, we introduce an extension of quantum tomography to an open quantum system after reviewing the general approach to a tomographic system [30, 31, 33].

Let us consider a quantum system that consists of $N$ degrees of freedom described with commuting Hermitian operators $\hat{A}_{1}, \hat{A}_{2}, \ldots . \hat{A}_{N}$. We describe a quantum state of the system with a density operator $\rho$ that is a trace class and Hermitian operator, such as

$$
\operatorname{Tr} \rho=1, \quad \rho^{\dagger}=\rho,
$$

and satisfies positivity which means the eigenvalues of $\rho$ are non-negative [34]. This enables us to interpret the diagonal elements of $\rho$ as a probability, expressed as

$$
\langle a| \rho|a\rangle \geqslant 0
$$

where $|a\rangle$ is a set of vector basis which consists of the eigenstates of the operators, $\hat{A}_{k}$, with $1 \leqslant k \leqslant N$ [31].

Using a Hermitian projection operator

$$
\begin{equation*}
\hat{\Pi}_{a}=|a\rangle\langle a|, \tag{1}
\end{equation*}
$$

we extract a state $|a\rangle$ and describe the probability of finding the system in that state as

$$
\begin{equation*}
\rho_{a, a}=\operatorname{Tr}\left[\hat{\Pi}_{a} \rho\right] . \tag{2}
\end{equation*}
$$

The transformation of the projection operator with a unitary operator $U(\xi)$

$$
\begin{equation*}
\hat{\Pi}_{a}(\xi)=U^{\dagger}(\xi)|a\rangle\langle a| U(\xi), \tag{3}
\end{equation*}
$$

where $\xi$ is a set of parameters to describe the unitary operation, gives the probability in the form

$$
\begin{align*}
\rho_{a, a}(\xi) & =\operatorname{Tr}\left[\hat{\Pi}_{a}(\xi) \rho\right]=\langle a| U(\xi) \rho U^{\dagger}(\xi)|a\rangle \\
& \equiv w(a, \xi) \tag{4}
\end{align*}
$$

which is called a tomogram. The tomogram enables us to describe a quantum state while preserving the quantum property. When treating a spin system, $U(\xi)$ corresponds to the rotation operators expressed using Euler angles (equation (8)), and hence the spin tomogram describes the probability in a rotated axis.

With the inversion of equation (4) such as

$$
\begin{equation*}
\rho=\int \mathrm{d} a \int \mathrm{~d} \xi w(a, \xi) K(a, \xi) \tag{5}
\end{equation*}
$$

we can reconstruct the original quantum state with the tomogram. For this purpose, we need to obtain a family of operators $K(a, \xi)$ that enable an unknown quantum state to be determined from the tomogram.

Next, let us consider the description of an open quantum dynamics system for a tomographic image. For this purpose, we consider that the relevant quantum system under measurement interacts with its environment. Defining the quantum state of the total system as $W$, which includes both the quantum system and its environment, and considering that the projective measurement in a transformed reference is performed only for the relevant system, the tomogram for the relevant system can be defined as

$$
\begin{align*}
w(a, \xi) & =\operatorname{Tr}\left[\left(\hat{\Pi}_{a}(\xi) \otimes I_{\mathrm{E}}\right) W\right]=\operatorname{Tr}_{\mathrm{S}}\left[\hat{\Pi}_{a}(\xi)\left(\operatorname{Tr}_{\mathrm{E}} W\right)\right] \\
& =\langle a| U(\xi) \rho_{\mathrm{S}} U^{\dagger}(\xi)|a\rangle \tag{6}
\end{align*}
$$

where $\mathrm{Tr}, \mathrm{Tr}_{\mathrm{S}}$ and $\mathrm{Tr}_{\mathrm{E}}$ denote the trace operations over the total system, the relevant quantum system and its environment, respectively, and $I_{\mathrm{E}}$ is the identity operator for the environment. In equation (6), the quantum state being measured can be treated as a reduced system obtained by averaging over the environmental state and can be defined as $\rho_{\mathrm{S}}=\operatorname{Tr}_{\mathrm{E}} W$. While the tomogram for an open system is described with a dynamical map [22, 24] and relaxation process under a Markovian approximation [25, 26], equation (6) enables us to reconsider these tomograms from a unified viewpoint that includes non-Markovian dynamics.

In the $\mu \mathrm{SR}$ experiments, we detect positrons emitted from muons that are injected into a sample material [32]. Since the
positrons are preferentially emitted along the direction of muon spin, the detection of positrons tells us the dynamics of muon spin which are under influence of its surroundings in the material. Such situations show a good correspondence to that described in equation (6) when we apply it to spin tomograms as is shown in the next section.

## 3. Spin tomography

Let us consider a state of spin with magnitude $j$ under the influence of environmental effects using the spin tomogram devised in [13-17, 20-22]. Defining an operator of the spin as $\hat{S}$, which has a $z$-component of $\hat{S}_{z}$, and defining the simultaneous eigenstate of $\hat{S}_{z}$ and $\hat{S}^{2}$ as $|j, m\rangle$, we obtain

$$
\begin{equation*}
\hat{S}_{z}|j, m\rangle=m|j, m\rangle, \quad \hat{S}^{2}|j, m\rangle=j(j+1)|j, m\rangle \tag{7}
\end{equation*}
$$

which gives the probability of finding the spin along the $z$ axis as $\rho_{\mathrm{S}, m, m}=\langle j, m| \rho_{\mathrm{S}}|j, m\rangle$. In spin tomography, the unitary transformation of the projection operator rotates the axes of the measurement, which we describe with Euler angles $\phi, \theta$ and $\psi$ as

$$
\begin{equation*}
U(\xi)=R(\phi, \theta, \psi)=\mathrm{e}^{\mathrm{i} \psi \hat{S}_{z}} \mathrm{e}^{\mathrm{i} \theta \hat{S}_{\mathrm{Y}}} \mathrm{e}^{\mathrm{i} \phi \hat{S}_{z}}, \tag{8}
\end{equation*}
$$

where $\hat{S}_{y}$ is the $y$-component of the spin operator $\hat{S}$. Substituting equation (8) into (6), we obtain a tomogram that describes the probability of the projection of the state $\left\langle j, m_{1}\right\rangle$ under the influence of environmental parameters onto the rotated $z$-axis in the form

$$
\begin{align*}
w & \left(m_{1}, \phi, \theta, \psi\right) \\
& =\left\langle j, m_{1}\right| R(\phi, \theta, \psi) \rho_{\mathrm{S}} R^{\dagger}(\phi, \theta, \psi)\left|j, m_{1}\right\rangle \\
& =\sum_{m 1^{\prime}=-j}^{j} \sum_{m 2^{\prime}=-j}^{j} D_{m 1, m 1^{\prime}}^{(j)}(\phi, \theta, \psi) \rho_{\mathrm{S}, m 1^{\prime}, m 2^{\prime}} D_{m 1, m 2^{\prime}}^{(j) *}(\phi, \theta, \psi), \tag{9}
\end{align*}
$$

where $D_{m i, m 1^{\prime}}^{(j)}(\phi, \theta, \psi)$ is the Wigner $D$ function, which is defined as

$$
\begin{equation*}
D_{m}^{(j)}, m(\phi, \theta, \psi)=\mathrm{e}^{\mathrm{i} m^{\prime} \psi} d_{m^{\prime} m}^{(j)}(\theta) \mathrm{e}^{\mathrm{i} m \phi} \tag{10}
\end{equation*}
$$

where

$$
\begin{align*}
d_{m^{\prime} m}^{(j)}(\theta)= & {\left[\frac{\left(j+m^{\prime}\right)!\left(j-m^{\prime}\right)!}{(j+m)!(j-m)!}\right]^{1 / 2}\left(\cos \frac{\theta}{2}\right)^{m^{\prime}+m} } \\
& \times\left(\sin \frac{\theta}{2}\right)^{m^{\prime}-m} P_{j-m^{\prime}}^{\left(m^{\prime}-m, m^{\prime}+m\right)}(\cos \theta), \tag{11}
\end{align*}
$$

with the Jacobi polynomial $P_{n}^{(a, b)}(x)$ [35]. Using the complex conjugation property of the Wigner $D$ function,

$$
D_{m 1, m 2^{\prime}}^{(j) *}(\phi, \theta, \psi)=(-1)^{m 1-m 2^{\prime}} D_{-m 1,-m 2^{\prime}}^{(j)}(\phi, \theta, \psi)
$$

in equation (9), we find that the angle $\psi$ is not necessary to define the spin tomogram. In the following, we omit $\psi$ from $w\left(m_{1}, \phi, \theta, \psi\right)$.

It has been shown that the tomogram can be inverted to reconstruct its corresponding density operator using the


Figure 1. 3D representation of the tomogram for the case of $\theta_{0}=\phi_{0}=0$ in equation (14).
expression

$$
\rho=\sum_{m_{1}=-j}^{j} \frac{1}{4 \pi} \int_{0}^{2 \pi} \mathrm{~d} \phi \int_{0}^{\pi} \mathrm{d} \theta w\left(m_{1}, \phi, \theta\right) \hat{D}\left(m_{1}, \phi, \theta\right)
$$

where $\hat{D}\left(m_{1}, \phi, \theta\right)$ is called the quantized operator, which has been defined in several ways using the Wigner $D$ function [14], a dynamical Lie group [31], and the star product [22]. Moreover, the reduction of the reconstruction with the integration over the $4 \pi$-steradian full solid angle into a finite number of rotations is proposed in [17], which requires the probabilities associated with only the three directions for a spin $\frac{1}{2}$ system. In the following, we focus on the spin tomogram, $w\left(m_{1}, \phi, \theta\right)$, which enables the spin state to be represented in the form of a probability function.

For a spin $\frac{1}{2}$ system, the tomogram is given by

$$
\begin{align*}
& w\left(\frac{1}{2}, \phi, \theta\right) \\
& \quad=\cos ^{2}\left(\frac{\theta}{2}\right) \rho_{\mathrm{S}, \frac{1}{2}, \frac{1}{2}}+\sin ^{2}\left(\frac{\theta}{2}\right) \rho_{\mathrm{S},-\frac{1}{2},-\frac{1}{2}} \\
& \quad+\sin \left(\frac{\theta}{2}\right) \cos \left(\frac{\theta}{2}\right)\left\{\mathrm{e}^{\mathrm{i} \phi} \rho_{\mathrm{S}, \frac{1}{2},-\frac{1}{2}}+\mathrm{e}^{-\mathrm{i} \phi} \rho_{\mathrm{S},-\frac{1}{2}, \frac{1}{2}}\right\} \tag{12}
\end{align*}
$$

and $\quad w\left(-\frac{1}{2}, \phi, \theta\right)=1-w\left(\frac{1}{2}, \phi, \theta\right)$. The tomogram $w\left(\frac{1}{2}, \phi, \theta\right)$ corresponds to the diagonal elements of a density operator for a spin $\frac{1}{2}$ system with an atomic coherent state [36] (the inverse of which is discussed in [37]) and with a spin coherent state [38-40].

When we consider a pure state as $\rho_{\mathrm{S}}=|\varphi\rangle\langle\varphi|$ for a superposed state $|\varphi\rangle$ written as

$$
\begin{equation*}
|\varphi\rangle=\cos \left(\frac{\theta_{0}}{2}\right)\left|\frac{1}{2}, \frac{1}{2}\right\rangle+\sin \left(\frac{\theta_{0}}{2}\right) \mathrm{e}^{\mathrm{i} \phi_{0}}\left|\frac{1}{2}, \frac{-1}{2}\right\rangle \tag{13}
\end{equation*}
$$

we obtain

$$
\rho_{\mathrm{S}}=\left(\begin{array}{cc}
\cos ^{2} \frac{\theta_{0}}{2} & \cos \frac{\theta_{0}}{2} \sin \frac{\theta_{0}}{2} \mathrm{e}^{-\mathrm{i} \phi_{0}}  \tag{14}\\
\cos \frac{\theta_{0}}{2} \sin \frac{\theta_{0}}{2} \mathrm{e}^{\mathrm{i} \phi_{0}} & \sin ^{2} \frac{\theta_{0}}{2}
\end{array}\right)
$$

In figure 1 , we show a plot of $w\left(\frac{1}{2}, \phi, \theta\right)$ for $\rho_{\mathrm{S}}$ with $\theta_{0}=\phi_{0}=0$ in equation (14) as a 3D representation of a curved surface. The distance between the origin and each point on the surface is the probability observed in the rotated
reference with the polar angle $(\theta, \phi)$ of the corresponding point on the surface.

There is a small probability in the region of $\frac{\pi}{2} \leqslant \theta<\pi$, which means that a near-opposite direction may be found in a pure state $\rho_{\mathrm{S}}=\left|\frac{1}{2}, \frac{1}{2}\right\rangle\left\langle\frac{1}{2}, \frac{1}{2}\right|$ as the projection of $m=\frac{1}{2}$ onto the rotated reference frame in $(\theta, \phi)$. This figure clearly shows that the spin tomogram can describe the quantum nature of a density operator.

In the next section, we apply the obtained spin tomogram to an open system where the time evolution of the spin $\frac{1}{2}$ system is described with the spin-boson model.

## 4. Application to a spin-boson model

Let us consider the dynamics of an open system, which is described by a relevant spin $\frac{1}{2}$ system that interacts with its environment. We consider that the environment consists of an infinite number of bosons. As an example of the spin-boson model, we take a system with a total Hamiltonian $\mathcal{H}$ of

$$
\begin{equation*}
\mathcal{H}=\mathcal{H}_{\mathrm{S}}+\mathcal{H}_{\mathrm{E}}+\mathcal{H}_{\mathrm{SE}} \tag{15}
\end{equation*}
$$

where $\mathcal{H}_{\mathrm{S}}$ is the relevant $\operatorname{spin} \frac{1}{2}$ system, $\mathcal{H}_{\mathrm{E}}$ represents the environment and $\mathcal{H}_{\mathrm{SE}}$ denotes the system-environment interaction. These terms are defined as

$$
\begin{align*}
\mathcal{H}_{\mathrm{S}} & =\sum_{m=0,1} \varepsilon_{m}|m\rangle\langle m|, \quad \mathcal{H}_{\mathrm{E}}=\sum_{k} \hbar \omega_{k} b_{k}^{\dagger} b_{k}, \\
\mathcal{H}_{\mathrm{SE}} & =\left\{\sum_{k} \hbar g_{k}\left(b_{k}^{\dagger}+b_{k}\right)\right\}(|0\rangle\langle 1|+|1\rangle\langle 0|), \tag{16}
\end{align*}
$$

where $|0\rangle=\left|\frac{1}{2},-\frac{1}{2}\right\rangle$ and $|1\rangle=\left|\frac{1}{2}, \frac{1}{2}\right\rangle$, and $b_{k}^{\dagger}$ and $b_{k}$ are the creation and annihilation boson operators of the $k$ th mode of the environment, and $\varepsilon_{1}-\varepsilon_{0}=\hbar \omega_{0}$.

The time evolution of the density operator for the total system, $W(t)$, is described with the Liouville-von Neumann equation as

$$
\begin{equation*}
\mathrm{i} \hbar \frac{\mathrm{~d}}{\mathrm{~d} t} W(t)=[\mathcal{H}, W(t)] . \tag{17}
\end{equation*}
$$

To obtain the dynamics of the relevant system $\rho_{\mathrm{S}}(t)=\operatorname{Tr}_{\mathrm{E}} W(t)$ under the system-environment interaction, we use the generalized master equation approach.

We assume that the initial condition of the total system can be described by a factorized form comprising the relevant system and the environment as $W(0)=\rho_{\mathrm{S}}(0) \rho_{\mathrm{E}}$, where we define $\rho_{\mathrm{S}}(0)$ and $\rho_{\mathrm{E}}$ as the initial states of the relevant system and the environment, respectively. We set $\rho_{\mathrm{E}}$ to be in Gibbs states such as $\rho_{\mathrm{E}}=\frac{1}{Z_{\mathrm{E}}} \exp \left[-\beta_{\mathrm{E}} \mathcal{H}_{\mathrm{E}}\right]$, where we define the partition functions as $\mathrm{Z}_{\mathrm{E}}=\operatorname{Tr}_{\mathrm{E}} \exp \left[-\beta_{\mathrm{E}} \mathcal{H}_{\mathrm{E}}\right]$ with an inverse temperature $\beta_{\mathrm{E}}=\frac{1}{k_{\mathrm{B}} T}$ for the Boltzmann constant $k_{\mathrm{B}}$. In general, the assumption of the factorized initial condition is justified for weak system-environment interactions. However, in the $\mu \mathrm{SR}$ experiment, the factorized initial condition represents the situation where the muons are initially injected into a material.

We use the time-convolutionless type of generalized master equation [41-44] in the form

$$
\begin{align*}
& \frac{\mathrm{d}}{\mathrm{~d} t} \rho_{\mathrm{S}}(t) \\
&=-\frac{\mathrm{i}}{\hbar}\left[\mathcal{H}_{\mathrm{S}}, \rho_{\mathrm{S}}(t)\right] \\
&+\int_{0}^{t} \mathrm{~d} \tau \operatorname{Tr}_{\mathrm{E}}\left[\left(\mathrm{i} \mathcal{L}_{\mathrm{SE}}(0)\right)\left(\mathrm{i} \mathcal{L}_{\mathrm{SE}}(-\tau)\right)\left(\rho_{\mathrm{E}} \rho_{\mathrm{S}}(t)\right)\right] \tag{18}
\end{align*}
$$

where $\mathcal{L}_{\mathrm{SE}}(t)$ is defined as

$$
\begin{equation*}
\mathcal{L}_{\mathrm{SE}}(t) A=\frac{1}{\hbar}\left[\mathcal{H}_{\mathrm{SE}}(t), A\right], \tag{19}
\end{equation*}
$$

for an arbitrary operator $A$, and $\mathcal{H}_{\mathrm{SE}}(t)$ is the Heisenberg picture of $\mathcal{H}_{\text {SE }}$, which is defined as

$$
\begin{equation*}
\mathcal{H}_{\mathrm{SE}}(t)=\mathrm{e}^{(\mathrm{i} / \hbar) \mathcal{H}_{0} t} \mathcal{H}_{\mathrm{SE}} \mathrm{e}^{-(\mathrm{i} / \hbar) \mathcal{H}_{0} t} \tag{20}
\end{equation*}
$$

with $\mathcal{H}_{0}=\mathcal{H}_{\mathrm{S}}+\mathcal{H}_{\mathrm{E}}$. Note that equation (18) is written in the original picture and not in the interaction picture.

In equation (18), we take up to the second order of the 'ordered' cumulant. The first 'ordered' cumulant vanishes because $\operatorname{Tr}_{\mathrm{E}} \rho_{\mathrm{E}} \mathcal{H}_{\mathrm{SE}}=0$ for this model, as expressed in equation (16).

We transform the reduced density operator $\rho_{\mathrm{S}}(t)$ into a vector using $\left|\rho_{\mathrm{S}}(t)\right\rangle=\left(\rho_{\mathrm{S}, 00}(t), \rho_{\mathrm{S}, 01}(t), \rho_{\mathrm{S}, 10}(t), \rho_{\mathrm{S}, 11}(t)\right)^{T}$ [45], which gives [46]

$$
\begin{equation*}
\left|\rho_{\mathrm{S}}(t)\right\rangle=T_{+} \exp \left[\int_{0}^{t} \mathrm{~d} t^{\prime} \boldsymbol{\Xi}\left(t^{\prime}\right)\right]\left|\rho_{\mathrm{S}}(0)\right\rangle \tag{21}
\end{equation*}
$$

with
$\Xi(t)=\Xi_{s}$

$$
-\int_{0}^{t} \mathrm{~d} \tau\left[\begin{array}{cccc}
V_{+}(\tau) & 0 & 0 & -V_{-}(\tau)  \tag{22}\\
0 & Y_{+}(\tau) & -Y_{-}(\tau) & 0 \\
0 & -Y_{+}(\tau) & Y_{-}(\tau) & 0 \\
-V_{+}(\tau) & 0 & 0 & V_{-}(\tau)
\end{array}\right] .
$$

In equation (22), $\Xi_{s}$ is a diagonal matrix with diagonal elements of $\left[0, \mathrm{i} \omega_{0},-\mathrm{i} \omega_{0}, 0\right]$ and

$$
\begin{gather*}
V_{ \pm}(\tau)=\left\{\Phi(\tau) \mathrm{e}^{\mathrm{Fi} \omega_{0} \tau}+\Phi(-\tau) \mathrm{e}^{\mathrm{ii} \omega_{0} \tau}\right\},  \tag{23}\\
Y_{ \pm}(\tau)=2 \mathfrak{R}(\Phi(\tau)) \mathrm{e}^{\mathrm{Fi} \omega_{0} \tau}, \tag{24}
\end{gather*}
$$

with

$$
\begin{equation*}
\Phi(\tau)=\sum_{k} g_{k}^{2}\left(\left\langle b_{k}^{\dagger} b_{k}\right\rangle \mathrm{e}^{\mathrm{i} \omega_{k} \tau}+\left\langle b_{k} b_{k}^{\dagger}\right\rangle \mathrm{e}^{-\mathrm{i} \omega_{k} \tau}\right) . \tag{25}
\end{equation*}
$$

Equation (21) demonstrates that the time dependence of the diagonal and off-diagonal elements of the density operator are decoupled. Moreover, equation (21) is an extension of equation (31) in [26], which can be used to describe nonMarkovian dynamics.

When we use continuous spectral density for the coupling strength $g_{k}$ in equation (25) as

$$
\begin{align*}
& h(\omega) \equiv \sum_{k} g_{k}^{2} \delta\left(\omega-\omega_{k}\right), \text { we obtain } \\
& \Phi(\tau) \\
& \quad=\int_{0}^{\infty} \mathrm{d} \omega h(\omega)\left\{n(\omega) \mathrm{e}^{\mathrm{i} \omega \tau}+(1+n(\omega)) \mathrm{e}^{\mathrm{-} \omega \tau}\right\} \\
& \quad=\int_{0}^{\infty} \mathrm{d} \omega h(\omega)\{(1+2 n(\omega)) \cos (\omega \tau)-\mathrm{i} \sin (\omega \tau)\}, \tag{26}
\end{align*}
$$

with $n(\omega)=1 /\left(\mathrm{e}^{\beta \hbar \omega}-1\right)$.
Using the time evolution of the reduced density operator described by equation (21) in equation (12) with the correspondence as $\rho_{\mathrm{S}, \frac{1}{2}, \frac{1}{2}} \leftrightarrow \rho_{\mathrm{S}, 11}(t), \rho_{\mathrm{S},-\frac{1}{2},-\frac{1}{2}} \leftrightarrow \rho_{\mathrm{S}, 00}(t)$ and so on, we obtain the dynamics for a tomogram for an open quantum system.

When we set the upper limit of the integral in the second term in equation (27) to infinity, we obtain the semi-group type of generator, the so-called Gorini-Kossakowski-Sudar-shan-Lindblad (GKSL) form [47]. In the next section, we numerically evaluate the dynamics of the elements of the density operator and tomogram by describing $h(\omega)$ as the Ohmic spectral density for both the non-Markovian and GKSL (Markovian) cases.

## 5. Numerical evaluation of spin tomogram

Let us consider the Ohmic spectral density with an exponential cutoff, $h(\omega)=s \omega \exp \left[-\omega / \omega_{c}\right]$, where $s$ is the coupling strength and $\omega_{c}$ is the cutoff frequency. In this case, an analytic form of $\Phi(\tau)$ can be expressed as

$$
\begin{align*}
& \Phi(\tau) \\
& =s\left\{\frac{\omega_{c}^{2}\left(1-\mathrm{i} \tau \omega_{c}\right)^{2}}{\left(1+\tau^{2} \omega_{c}^{2}\right)^{2}}+\frac{1}{\beta^{2}} \mathfrak{R}\left(\psi^{\prime}\left(1+\frac{1}{\beta \omega_{c}}-\mathrm{i} \frac{\tau}{\beta}\right)\right)\right\}, \tag{27}
\end{align*}
$$

with $\psi^{\prime}(z)=\frac{\mathrm{d}}{\mathrm{d} z} \psi(z)$ where $\psi(z)$ is a polygamma function defined in terms of the Euler Gamma function $\Gamma(z)$ as $\psi(z)=\frac{\mathrm{d}}{\mathrm{d} z} \Gamma(z) / \Gamma(z)$.

Substituting equation (27) in (21), we obtain the time evolution of $\rho_{00}(t)$ and $\rho_{01}(t)$. We scale the time variable as $\tilde{t}=\omega_{0} t$ and the cutoff frequency as $\tilde{\omega}_{c}=\omega_{c} / \omega_{0}$ and set $s=0.01$, which means a weak system-environment interaction. In figure 2 , the time evolution of $\rho_{00}(\tilde{t})$ is shown for an initial condition of a pure state determined by $\theta_{0}=\phi_{0}=\frac{3 \pi}{4}$ in equation (14) and for the environmental temperature $T$ determined as $k_{\mathrm{B}} T=5 \hbar \omega_{0}$.

Increasing $\tilde{\omega}_{c}$ from 0.3 to 3 for the non-Markovian cases (solid lines in figure 2), the behavior changes from oscillatory to monotonic, with both cases asymptotically approaching a stationary value determined by the environmental temperature. Because the cutoff frequency $\omega_{c}$ corresponds to the reciprocal of the correlation time of the system-environment interaction, the oscillation for $\tilde{\omega}_{c}=0.3$ indicates back action from the environment because of the long system-environment correlation time. The Markovian GKSL cases (for $\tilde{\omega}_{c}=0.3$ and 3 ) are given in the dashed lines in figure 2 , both


Figure 2. Time evolution of $\rho_{00}(\tilde{t})$ for $s=0.01$ with changing $\tilde{\omega}_{c}=0.3$ and $\tilde{\omega}_{c}=3$ for an initial condition of a pure state determined by $\theta_{0}=\phi_{0}=\frac{3 \pi}{4}$ in equation (14) and for the environmental temperature $T$ determined as $k_{\mathrm{B}} T=5 \hbar \omega_{0}$. The solid and dashed lines refer respectively to the non-Markovian cases and those obtained with the GKSL form.


Figure 3. Time evolution of $\mathfrak{R}\left(\rho_{01}(\tilde{t})\right)$ and $\mathfrak{I}\left(\rho_{01}(\tilde{t})\right)$ for $\tilde{\omega}_{c}=0.3$. All other parameters are the same as in figure 2. The solid and dashed lines refer respectively to the non-Markovian cases and those obtained with the GKSL form.
of which exhibit monotonic decay. The difference between the non-Markovian and Markovian cases for $\tilde{\omega}_{c}=0.3$ is larger than that for $\tilde{\omega}_{c}=3$. Because the Markovian GKSL form is obtained on the assumption that the system-environment correlation time is infinitely short, this feature is reasonable.

Figure 3 shows the time evolution of $\mathfrak{R}\left(\rho_{01}(\tilde{t})\right)$ and $\mathfrak{I}\left(\rho_{01}(\tilde{t})\right)$ for $\tilde{\omega}_{c}=0.3$ with the same parameters as in figure 2. The solid lines show the non-Markovian cases and the dashed lines refer to those for the GKSL form. For both cases, the oscillation of the off-diagonal element persists for a long time because of the long system-environment correlation time as determined by $\tilde{\omega}_{c}=0.3$. The amplitude of the oscillation for the non-Markovian cases undergoes a faster decay than for the Markovian cases, and thus reflects the decay occurring in the diagonal elements.


Figure 4. Time evolution of $\mathfrak{R}\left(\rho_{01}(\tilde{t})\right)$ and $\mathfrak{I}\left(\rho_{01}(\tilde{t})\right)$ for $\tilde{\omega}_{c}=3$. All other parameters are the same as in figure 2. The solid and dashed lines refer respectively to the non-Markovian cases and those obtained with the GKSL form.


Figure 5. Time evolution of a tomogram $w\left(\frac{1}{2}, \phi, \theta\right)$ for $\tilde{\omega}_{c}=0.3$ at $\tilde{t}=1,3,5,7$ for the non-Markovian cases. All other parameters are the same as in figure 2 .

In figure 4 , we show the time evolution of $\mathfrak{R}\left(\rho_{01}(\tilde{t})\right)$ and $\Im\left(\rho_{01}(\tilde{t})\right)$ for $\tilde{\omega}_{c}=3$ where all other parameters are the same as in figure 2. The solid lines show the non-Markovian cases and the dashed lines refer those for the Markovian GKSL form. For both cases, the oscillations of the off-diagonal element decay with time because of the short system-environment correlation time corresponding to $\tilde{\omega}_{c}=3$. The evaluations for the Markovian form show a faster decay than for the non-Markovian cases which corresponds to the feature in the diagonal elements.

In figure 5, we show the time evolution of a tomogram $w\left(\frac{1}{2}, \phi, \theta, \psi\right)$ for $\tilde{\omega}_{c}=0.3$ at $\tilde{t}=1,3,5,7$ for non-Markovian cases. All other parameters are the same as in figure 2.


Figure 6. The tomograms for the non-Markovian and Markovian cases for $\tilde{\omega}_{c}=0.3$ at $\tilde{t}=3$. The left (right) figure shows the nonMarkovian (Markovian) cases. All other parameters are the same as in figure 2.


Figure 7. Time evolution of the tomogram $w\left(\frac{1}{2}, \phi, \theta\right)$ for $\tilde{\omega}_{c}=3$ at $\tilde{t}=1,3,5,7$ for the non-Markovian cases. All other parameters are the same as in figure 2.

The oscillation of $\rho_{00}(\tilde{t})$ and $\rho_{01}(\tilde{t})$, which persists for a long time as shown in figures 2 and 3, indicates that the tomogram rotates in this time range while retaining its asymmetrical shape.

Figure 6 gives tomograms for the non-Markovian and Markovian cases for $\tilde{\omega}_{c}=0.3$ at $\tilde{t}=3$. Comparing these tomograms, we find the probability of finding positive values is slightly larger for the Markovian case than for the nonMarkovian case. This reflects the time evolution of the elements of the density operator shown in figures 2 and 3. Tomograms for the other Markovian and non-Markovian cases are very similar (figure 5).

In figure 7, the time evolution of the tomogram $w\left(\frac{1}{2}, \phi, \theta\right)$ for the non-Markovian case is shown when $\tilde{\omega}_{c}$ is increased to 3 for $\tilde{t}=1,3,5,7$ with the same parameters as
in figure 2 . In contrast to the case with $\tilde{\omega}_{c}=0.3$ shown in figure 5, the tomogram loses its asymmetrical shape with time as the relevant system approaches a stationary state. The tomograms for the Markovian and non-Markovian cases (figure 7) are very similar as expected from the time evolution of the elements of density operator (figures 2 and 4 ).

## 6. Concluding remarks

In this paper, we developed a way to describe the quantum tomography of an open system using a generalized master equation. We applied this to a spin tomogram, considering a case where the relevant spin under tomographic measurement suffers inevitable effects from the environment. Describing such effects with a spin-boson model where the relevant spin interacts with a bosonic environment, we obtained a timeconvolutionless type of generalized master equation. We transformed the obtained master equation into the HilbertSchmidt space and found an analytic form of the supermatrix, which determined the time evolution of the density operator of the spin for the Ohmic spectral density with an exponential cutoff. With the obtained supermatrix, we numerically evaluated the time evolution of elements of the density operator and spin tomogram, which showed clear interdependency.

Here we mention the relationship between the spin tomogram $w\left(m_{1}, \theta, \phi\right)$ and the $\mu \mathrm{SR}$ experiments. If we can place the positron detectors so as to cover the $4 \pi$ steradian in measuring an ensemble of muons, we could obtain the tomogram of the spin state, which enables the reconstruction of $\rho$ by averaging a quantizer operator over the tomogram as discussed in section 3. However, such an experimental setup is challenging as discussed in [23]. To overcome such difficulties, it has been reported in [17] that the reconstruction is possible with a finite number of rotations which requires the probabilities in only three directions in measuring the spin $\frac{1}{2}$ system. Such reconstructions share similar concepts found with the method discussed in [48]. As these methods require the measurements of the spin state in finite directions, to increase the accuracy of reconstruction of $\rho$, we need a greater number of measurements. The graphical representation of the probability function of spin state as discussed in this paper provides the detailed physical background for the need for such a great number of measurements. We believe this formulation will provide the fundamental information for the analyses of experimental techniques such as $\mu \mathrm{SR}$ over short time scales and/or at low temperatures.

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