Physica Scripta

PAPER

Tomography of a quantum state related to the Husimi function

To cite this article: Ye-jun Xu *et al* 2012 *Phys. Scr.* **85** 025014

IOP PUBLISHING PHYSICA SCRIPTA

Phys. Scr. **85** (2012) 025014 (4pp) doi:10.1088/0031-8949/85/02/025014

Tomography of a quantum state related to the Husimi function

Ye-jun Xu, Shu-dong Fang, Xue-ping Zang and Chun Miao

Department of Physics Electronic and Engineering, Chizhou University, Chizhou, Anhui 247000, People's Republic of China

E-mail: yejunxu@126.com

Received 18 October 2011
Accepted for publication 20 December 2011
Published 23 January 2012
Online at stacks.iop.org/PhysScr/85/025014

Abstract
We establish a new integral transform that reveals the relation of the density operator to the Husimi operator. The connection between the Husimi function of a given state (i.e. position–momentum-intermediate representation) and the tomography of an arbitrary quantum state is revealed.

PACS numbers: 03.65.Wj, 02.30.Uu, 42.30.Wb, 42.50.Dv

1. Introduction

In conventional quantum mechanics, usually we use a state vector in the Hilbert space or a density operator to depict a quantum system. As an alternative approach, a direct description of quantum states by means of positive probability-distribution functions for system observables is of interest from both the theoretical and the experimental point of view. For a long time, physicists have been trying to find such a classical-like description of quantum states and avoid the notion of the wave function. Although the Heisenberg uncertainty relation for the position and momentum of quantum systems tells us that there does not exist an actual joint probability-distribution function in the phase space of quantum mechanics, some kinds of classical-like descriptions of quantum states (i.e. quasi-probability distributions) have still been introduced. These quasi-probability distributions not only are useful as computational tools, but also provide insights into the connection between classical and quantum mechanics. These quasi-probability distributions allow one to express quantum mechanical averages in a form similar to classical averages. The prominent ones among these quasi-probability functions are the Wigner function [1, 2], the Husimi Q-function [3] and the Glauber–Sudarshan P-function [4, 5]. They have been widely used as instruments for calculations in quantum optics.

In recent years, quantum tomography has become a useful tool for the recovery of the quantum state from a set of measured probability distributions. In the context of phase-space theory of quantum statistical physics, Vogel and Risken [6] pointed out that the probability distribution for the rotated quadrature phase can be expressed in terms of the Wigner function. In quantum optics theory, all possible linear combinations of the field quadratures $\mu Q+\nu P$ can be measured by the homodyne measurement just by varying the phase of the local oscillator. The average of the measurements, at a given local oscillator phase, is related to the marginal distribution of the Wigner function. This method has been experimentally realized [7].

As is well known, the density operator can be expanded in terms of a complete set of operators; one can clarify different aspects of the quantum state by using different representations to construct the corresponding density matrix. Analogously, the case can also be seen in the tomographic approach [8–24]. For instance, in [9], a new tomographic representation (squeeze tomography), which is the photon number analogous to the case of photon-number tomographys, was introduced to study the relation of the squeeze tomograms to the symplectic tomograms. In [14], the authors realized the map of the s-parameterized quasi-distribution function to the corresponding quantum-state tomogram. In this work, based on the fact that the Husimi operator $\Delta_{\mathrm{h}}(q, p, \kappa)$ is just the squeezed coherent state projector [25, 26], i.e. $\Delta_{\mathrm{h}}(q, p, \kappa) = |p, q, \kappa\rangle\langle p, q, \kappa|$ and $|p, q, \kappa\rangle$ span a complete quantum mechanical representation, our main aim is to try to find the connection between the quantum-state tomogram and the Husimi function.

The paper is organized as follows. In section 2, we briefly review the knowledge about the Husimi operator. In section 3, we introduce a new expansion of the density operator in terms of the Husimi operator $\Delta_{\mathrm{h}}(q, p, \kappa)$. In section 4, we establish an integral transform that maps the Husimi function of the position–momentum-intermediate representation $|x\rangle_{\mu,\nu}$ to the corresponding quantum-state tomogram.

2. The Husimi operator as the squeezed coherent state projector

In order to overcome the inconvenience that the Wigner distribution function itself can attain negative values and fail to act as a probability distribution, the Husimi distribution function $F_{\mathrm{h}}(q, p, \kappa)$ was naturally introduced [3]. It was defined by smoothing out the Wigner function with the aid of averaging over a 'coarse graining' function

$$\begin{aligned} F_{\mathrm{h}}(q, p, \kappa) = & \iint_{-\infty}^{\infty} \mathrm{d}q'\mathrm{d}p'\Delta_{\mathrm{w}}\left(q', p'\right) \\ & \times \exp\left[-\kappa\left(q'-q'\right)^2 - \frac{(p'-p)^2}{\kappa}\right] \\ = & \operatorname{Tr}\left[\rho\Delta_{\mathrm{h}}\left(q, p, \kappa\right)\right], \end{aligned} \tag{1}$$

where κ is the Gaussian spatial width parameter, and $\Delta_{\mathrm{w}}(q, p)$ and $\Delta_{\mathrm{h}}(q, p, \kappa)$ are the Wigner and Husimi operators, respectively. Equation (1) reveals the connection between the Wigner function and the Husimi function. Accordingly, the relation between $\Delta_{\mathrm{h}}(q, p, \kappa)$ and $\Delta_{\mathrm{w}}(q, p)$ is

$$\begin{aligned} \Delta_{\mathrm{h}}(q, p, \kappa) = & \iint_{-\infty}^{\infty} \mathrm{d}q'\mathrm{d}p'\Delta_{\mathrm{w}}(q', p') \\ & \times \exp\left[-\kappa(q'-q')^2 - \frac{(p'-p)^2}{\kappa}\right]. \end{aligned} \tag{2}$$

It is worth noting that equation (2), according to the Weyl quantization scheme [27], shows that $\Delta_{\mathrm{h}}(q, p, \kappa)$ is the Weyl correspondence quantization operator of the so-called 'coarse graining' function $\mathrm{e}^{-\kappa(q'-q')^2-(p'-p)^2/\kappa}$. Furthermore, the Husimi operator $\Delta_{\mathrm{h}}(q, p, \kappa)$ is revealed to be just the squeezed coherent state projector [25, 26],

$$\Delta_{\mathrm{h}}(q, p, \kappa) = |p, q, \kappa\rangle\langle p, q, \kappa|, \tag{3}$$

in which $|p, q, \kappa\rangle$ is a kind of squeezed coherent state and has the form in the Fock space

$$\begin{aligned} |p, q, \kappa\rangle = & \left(\frac{2\sqrt{\kappa}}{1+\kappa}\right)^{1/2} \exp\left[\frac{-\kappa}{2(1+\kappa)}q^2 - \frac{p^2}{2(1+\kappa)}\right. \\ & \left. + \frac{\sqrt{2}a^{\dagger}}{1+\kappa}(\kappa q + \mathrm{i}p) + \frac{1-\kappa}{2(1+\kappa)}a^{\dagger 2}\right]|0\rangle \\ = & S^{-1}(\sqrt{\kappa})D(\alpha)|0\rangle \\ = & S^{-1}(\sqrt{\kappa})|\alpha\rangle \equiv |\alpha, \kappa\rangle, \end{aligned} \tag{4}$$

where $|\alpha\rangle$ is Glauber's coherent state, $\alpha = (\sqrt{\kappa}q + \mathrm{i}p/\sqrt{\kappa})/\sqrt{2}$, $S^{-1}(\sqrt{\kappa}) = \exp[(a^{\dagger 2} - a^2)\ln(1/\sqrt{\kappa})]$ is the usual squeezing operator and $D(\alpha) = \exp(\alpha a^{\dagger} - \alpha^* a)$ is the standard displacement operator. So κ in $|p, q, \kappa\rangle$ relates to both the displacement and the squeezing. According to

$$\int \frac{\mathrm{d}\alpha^2}{\pi}|\alpha, \kappa\rangle\langle\alpha, \kappa| = \int \frac{\mathrm{d}\alpha^2}{\pi}S^{-1}(\sqrt{\kappa})|\alpha\rangle\langle\alpha|S(\sqrt{\kappa}) = 1, \tag{5}$$

$|\alpha, \kappa\rangle$ is capable of making up a quantum mechanical representation.

3. New representation of the density operator via the Husimi operator

In this section, we present a new expansion formula about the density operator. Based on the completeness relation in equation (5), a density operator ρ can be expressed as

$$\rho = \int \frac{\mathrm{d}^2\alpha}{\pi}F(\alpha)|\alpha, \kappa\rangle\langle\alpha, \kappa|. \tag{6}$$

The function $F(\alpha)$ may be referred to as the squeezed coherent state representation; it represents the density operator ρ by means of a diagonal representation in terms of the squeezed coherent state. In the following, it is necessary to confine our attention to calculating $F(\alpha)$. Multiplying both sides of equation (6) by the coherent state $\langle -z|$ from the left and $|z\rangle$ from the right, we obtain

$$\langle -z|\rho|z\rangle = \int \frac{\mathrm{d}^2\alpha}{\pi}F(\alpha)\langle -z|\alpha, \kappa\rangle\langle\alpha, \kappa|z\rangle. \tag{7}$$

In view of equation (4) and the normally ordered form of the single-mode squeezing operator [28]

$$\begin{aligned} S(\sqrt{\kappa}) = & \sqrt{\operatorname{sech}(\ln\sqrt{\kappa})} : \exp\left[-\frac{a^{\dagger 2}}{2}\tanh(\ln\sqrt{\kappa})\right. \\ & \left. + (\operatorname{sech}(\ln\sqrt{\kappa}) - 1)a^{\dagger}a + \frac{a^2}{2}\tanh(\ln\sqrt{\kappa})\right]:, \end{aligned} \tag{8}$$

where the symbol :: denotes the normal ordering, we have

$$\begin{aligned} \langle\alpha, \kappa|z\rangle = & \sqrt{\operatorname{sech}(\ln\sqrt{\kappa})}\exp\left[-\frac{1}{2}|z|^2 + \frac{\tanh(\ln\sqrt{\kappa})}{2}z^2\right] \\ & \times \exp\left[-\frac{1}{2}|\alpha|^2 - \frac{\alpha^{*2}}{2}\tanh(\ln\sqrt{\kappa}) + \operatorname{sech}(\ln\sqrt{\kappa})\alpha^* z\right]. \end{aligned} \tag{9}$$

It then follows that

$$\begin{aligned} \langle -z|\alpha, \kappa\rangle\langle\alpha, \kappa|z\rangle = & \operatorname{sech}(\ln\sqrt{\kappa})\exp[\operatorname{sech}(\ln\sqrt{\kappa})(\alpha^* z \\ & - \alpha z^*)]\exp\left[-|\alpha|^2 - |z|^2 - \frac{\tanh(\ln\sqrt{\kappa})}{2}(\alpha^{*2} + \alpha^2)\right. \\ & \left. + \frac{\tanh(\ln\sqrt{\kappa})}{2}(z^2 + z^{*2})\right]. \end{aligned} \tag{10}$$

Substituting equation (10) into (7) and setting $\beta = \operatorname{sech}(\ln\sqrt{\kappa})\alpha$ yield

$$\begin{aligned} \langle -z|\rho|z\rangle = & \frac{1}{\operatorname{sech}(\ln\sqrt{\kappa})}\int \frac{\mathrm{d}^2\beta}{\pi}F\left(\frac{\beta}{\operatorname{sech}(\ln\sqrt{\kappa})}\right) \\ & \times \exp\left[-\frac{1}{(\operatorname{sech}(\ln\sqrt{\kappa}))^2}|\beta|^2 - \frac{\beta^{*2} + \beta^2}{2\tanh(\ln\sqrt{\kappa})}\right] \\ & \times \exp\left[-|z|^2 + \frac{\tanh(\ln\sqrt{\kappa})}{2}(z^2 + z^{*2})\right]\exp(\beta^* z - \beta z^*). \end{aligned} \tag{11}$$

In view of $\beta^* z - \beta z^*$ being a purely imaginary quantity, equation (11) is actually a two-dimensional Fourier-transform relation. On taking the Fourier inverse of equation (11), we obtain the expression of $F(\alpha)$ as follows:

$$F(\alpha) = \operatorname{sech}(\ln\sqrt{\kappa}) \exp\left[|\alpha|^2 + \frac{\tanh(\ln\sqrt{\kappa})}{2}(\alpha^{*2}+\alpha^2)\right] \times \int \frac{d^2 z}{\pi}\langle -z|\rho|z\rangle \exp\left[|z|^2 - \frac{\tanh(\ln\sqrt{\kappa})}{2}(z^2+z^{*2})\right] \times \exp\left[\operatorname{sech}(\ln\sqrt{\kappa})(\alpha z^* - \alpha^* z)\right]. \quad (12)$$

In particular, when $\kappa = 1$, $F(\alpha)$ becomes

$$F(\alpha) = \exp\left(|\alpha|^2\right) \int \frac{d^2 z}{\pi}\langle -z|\rho|z\rangle \exp\left(|z|^2 + z^*\alpha - z\alpha^*\right), \quad (13)$$

which is just the Glauber–Sudarshan P-representation [4, 5].

Note that when replacing the density operator ρ with an arbitrary operator A in equations (6) and (12), they can still hold, namely

$$A = \int \frac{d^2\alpha}{\pi} F(\alpha)\,|\alpha,\kappa\rangle\,\langle\alpha,\kappa|\,. \quad (14)$$

Correspondingly, the representation $F(\beta)$ reads

$$F(\alpha) = \operatorname{sech}(\ln\sqrt{\kappa}) \exp\left[|\alpha|^2 + \frac{\tanh(\ln\sqrt{\kappa})}{2}(\alpha^{*2}+\alpha^2)\right] \times \int \frac{d^2 z}{\pi}\langle -z|A|z\rangle \exp\left[|z|^2 - \frac{\tanh(\ln\sqrt{\kappa})}{2}(z^2+z^{*2})\right] \times \exp[\operatorname{sech}(\ln\sqrt{\kappa})(\alpha z^* - \alpha^* z)]. \quad (15)$$

4. Calculation of the quantum-state tomogram on the Husimi function

The tomographic symbol of a quantum state described by the density operator ρ can be defined in terms of its Wigner function, in view of the Radon transformation [29–33], as follows:

$$w(x,\mu,\nu) = \iint_{-\infty}^{\infty} dq\, dp\, \delta(x-\mu q-\nu p) W(q,p). \quad (16)$$

On the other hand, the Radon transform of the Wigner operator can be concisely calculated:

$$\int dx' dp' \delta(p-\mu x'-\nu p')\Delta(x',p') = \frac{1}{\sqrt{\pi(\mu^2+\nu^2)}}\colon \times \exp\left[-\frac{p^2}{\mu^2+\nu^2} + \frac{i\sqrt{2}pa^\dagger}{\nu+i\mu} - \frac{i\sqrt{2}pa}{\nu-i\mu} + \frac{(\nu-i\mu)a^{\dagger 2}}{2(\nu+i\mu)} + \frac{(\nu-i\mu)^2}{2(\nu-i\mu)} - a^\dagger a\right]\colon, \quad (17)$$

where we have used the normally ordered expression for the Wigner operator [34]

$$\Delta(x,p) = \frac{1}{\pi}\colon \exp\left[-(x-Q)^2-(p-P)^2\right]\colon. \quad (18)$$

Using the normally ordered form of the vacuum state projector $|0\rangle\langle 0| =\colon e^{-a^\dagger a}\colon$, one can decompose the right-hand side of equation (17) as

$$\int dx' dp' \delta(p-\mu x'-\nu p')\Delta(x',p') = |x\rangle_{\mu,\nu\;\mu,\nu}\langle x|\,, \quad (19)$$

where we introduced a new vector

$$|x\rangle_{\mu,\nu} = \frac{1}{\left[\pi(\mu^2+\nu^2)\right]^{1/4}} \exp\left(-\frac{1}{2(\mu^2+\nu^2)}x^2 + \frac{\sqrt{2}x}{\mu-i\nu}a^\dagger - \frac{\mu+i\nu}{2(\mu-i\nu)}a^{\dagger 2}\right)|0\rangle\,. \quad (20)$$

If $\mu=1$, $\nu=0$, and $\mu=0$, $\nu=1$, then $|x\rangle_{\lambda,\nu}$ reduces to the coordinate eigenvector

$$|x\rangle_{1,0} = \pi^{-1/4}\exp\left[-\frac{q^2}{2} + \sqrt{2}qa^\dagger - \frac{a^{\dagger 2}}{2}\right]|0\rangle \quad (21)$$

and the momentum eigenvector

$$|x\rangle_{0,1} = \pi^{-1/4}\exp\left[-\frac{p^2}{2} + \sqrt{2}ipa^\dagger + \frac{a^{\dagger 2}}{2}\right]|0\rangle\,, \quad (22)$$

respectively. Thus $|x\rangle_{\mu,\nu}$ is called the position–momentum-intermediate representation [35]. It satisfies the eigenstate equation

$$(\mu Q+\nu P)\,|x\rangle_{\mu,\nu} = x|x\rangle_{\mu,\nu} \quad (23)$$

and the completeness relation

$$\int_{-\infty}^{\infty} dx|x\rangle_{\mu,\nu\;\mu,\nu}\langle x| = \frac{1}{\sqrt{\pi(\mu^2+\nu^2)}}\int_{-\infty}^{\infty} dx\colon \times \exp\left[-\frac{(x-\mu Q-\nu P)^2}{\mu^2+\nu^2}\right]\colon = 1. \quad (24)$$

Equation (19) shows that the Radon transform of the Wigner operator is just the pure state density matrix $|x\rangle_{\mu,\,\nu\mu,\nu}\langle x|$. So the quantum tomogram of a quantum state, the Radon transform of the Wigner function, can also be read as

$$w(x,\mu,\nu) = \mathrm{Tr}(\rho|x\rangle_{\mu,\nu\;\mu,\nu}\langle x|) =_{\mu,\nu}\langle x|\rho|x\rangle_{\mu,\nu}, \quad (25)$$

and the tomographic operator $|x\rangle_{\mu,\nu\,\mu,\nu}\langle x|$ is just the projector of $|x\rangle_{\mu,\nu}$.

Again, the following transform relation has been shown in the literature [9]:

$$S(\lambda)\,R(\theta)\,QR^\dagger(\theta)\,S^\dagger(\lambda) = \mu Q+\nu P, \quad (26)$$

where $\mu = e^{\lambda}\cos\theta$, $\nu = e^{-\lambda}\sin\theta$, $R(\theta) = \exp\left[\frac{i\theta}{2}(Q^2+P^2)\right]$ is the rotation operator and $\mu Q+\nu P$ represents all possible linear combinations of quadratures Q and P of the oscillator field mode and can be measured by the homodyne measurement just by varying the phase of the local oscillator. $S(\lambda)$ is the squeezing operator with the squeezing parameter λ. Substituting equation (6) into (25) yields

$$w(x,\mu,\nu) = \int \frac{d^2\alpha}{\pi}F(\alpha)_{\mu,\nu}\langle x|\alpha,\kappa\rangle\langle\alpha,\kappa|x\rangle_{\mu,\nu} = \int \frac{d^2\alpha}{\pi}F(\alpha)|\langle\alpha,\kappa|x\rangle_{\mu,\nu}|^2, \quad (27)$$

where the integral kernel $|\langle\alpha,\kappa|x\rangle_{\mu,\nu}|^2$ is just the Husimi function of $|x\rangle_{\mu,\nu}$. Thus, we establish a connection of the quantum-state tomogram corresponding to a given density operator with the Husimi function of the position–momentum-intermediate representation $|x\rangle_{\mu,\nu}$.

We now refer mainly to calculating the explicit formula of the Husimi function $|\langle\alpha,\kappa|x\rangle_{\mu,\nu}|^2$. As a result of equation (4), one can rewrite $\langle\alpha,\kappa|x\rangle_{\mu,\nu}$ as

$$\langle\alpha,\kappa|x\rangle_{\mu,\nu} = \langle\alpha|S(\sqrt{\kappa}+\lambda)R(\theta)|x\rangle. \tag{28}$$

Let $\lambda' = \ln\sqrt{\kappa}+\lambda$; then $\mu' = e^{\lambda'}\cos\theta$, $\nu' = e^{-\lambda'}\sin\theta$ and equation (28) becomes

$$\langle\alpha,\kappa|x\rangle_{\mu,\nu} = \frac{1}{[\pi(\mu'^2+\nu'^2)]^{1/4}}\exp\left(-\frac{1}{2}|\alpha|^2 - \frac{1}{2(\mu'^2+\nu'^2)}x^2 + \frac{\sqrt{2}x}{\mu'-i\nu'}\alpha^* - \frac{\mu'+i\nu'}{2(\mu'-i\nu')}\alpha^{*2}\right). \tag{29}$$

The Husimi function $|\langle\alpha,\kappa|x\rangle_{\mu,\nu}|^2$ can therefore be written as

$$|\langle\alpha,\kappa|x\rangle_{\mu,\nu}|^2 = \frac{1}{\sqrt{\pi(\mu'^2+\nu'^2)}} \times\exp\left(-|\alpha|^2 - \frac{1}{(\mu'^2+\nu'^2)}x^2 + \frac{\sqrt{2}x}{\mu'-i\nu'}\alpha^*\right) \times\exp\left(\frac{\sqrt{2}x}{\mu'+i\nu'}\alpha - \frac{\mu'+i\nu'}{2(\mu'-i\nu')}\alpha^{*2} - \frac{\mu'-i\nu'}{2(\mu'+i\nu')}\alpha^2\right). \tag{30}$$

If we adopt the notation $\eta = (\mu'+i\nu')/\sqrt{2}$, then equation (30) can further be rewritten in a more concise form

$$|\langle\alpha,\kappa|x\rangle_{\mu,\nu}|^2 = \frac{1}{|\eta|\sqrt{2\pi}}\exp\left[-\frac{(x-\eta\alpha^*-\eta^*\alpha)^2}{2|\eta|^2}\right]. \tag{31}$$

In particular, when $\kappa = 1$, equation (27) is simplified to calculate the tomogram from the Glauber–Sudarshan P-function (please see also [14]). The difference between equation (31) and the standard definition of the Radon transform in equation (16) is only to replace an ideally sharp δ-function by a Gaussian with dispersion $|\eta|$ related to the parameters κ, λ, μ and ν.

5. Conclusion

By virtue of the fact that the Husimi operator is just the squeezed coherent state projector that possesses the completeness relation, we have established a new integral transform, which reveals the connection of an arbitrary operator, especially for the density operator ρ, to the Husimi operator $\Delta_h(q,p,\kappa)$. Furthermore, in view of the fact that the tomographic symbol of the density operator can be found from the corresponding Wigner function by the standard Radon transform and by means of the new integral transform about the density operator in equation (6), we have expressed the tomographic symbol of the density operator in terms of the Husimi function of the position–momentum-intermediate representation $|x\rangle_{\mu,\nu}$.

Acknowledgments

We sincerely thank the referee for his useful suggestions and Professors Xian-cai Wang, Ming-gao Wan and Chuan-you Gui for helpful discussions. This work is supported by the Key Program of the Education Department of Anhui Province (Grant No. KJ2011Z268), the Higher School Fund of Outstanding Young Talent (Grant No. 2010SQRL132) and the Scientific Research Foundation of Chizhou University (Grant No. 2010RC036).

References

[1] Wigner E P 1932 *Phys. Rev.* **40** 749
[2] Hillery M, Hillery M, O'Connell R F, Scully M O and Wigner E P 1984 *Phys. Rep.* **106** 121
[3] Husimi K 1940 *Proc. Phys. Math. Soc. Japan* **22** 264
[4] Sudarshan C G E 1963 *Phys. Rev. Lett.* **10** 177
[5] Glauber R J 1963 *Phys. Rev. Lett.* **10** 84
[6] Vogel K and Risken H 1989 *Phys. Rev.* **40** 2847
[7] Smithey D T, Beck M, Raymer M G and Faridani A 1993 *Phys. Rev. Lett.* **70** 1244
[8] Bazrafkan M R and Man'ko V I 2004 *J. Russ. Laser Res.* **25** 453
[9] Castaños O, López-Peña R, Man'ko M A and Man'ko V I 2004 *J. Phys. A: Math. Gen.* **37** 8529
[10] Man'ko M A 2000 *J. Russ. Laser Res.* **21** 411
[11] Mancini S, Man'ko V I, Shchukin E V and Tombesi P 2003 *J. Opt. B: Quantum Semiclass. Opt.* **5** S333
[12] Man'ko M A, Man'ko V I and Mendes R V 2001 *J. Phys. A: Math. Gen.* **34** 8321
[13] Man'ko O V, Man'ko V I and Marmo G 2002 *J. Phys. A: Math. Gen.* **35** 699
[14] Bazrafkan M R and Man'ko V I 2011 *J. Russ. Laser Res.* **32** 230
[15] Mancini S, Man'ko V I and Tombesi P 1997 *J. Mod. Opt.* **44** 2281
[16] Man'ko O V and Man'ko V I 2009 *Laser Phys.* **19** 1804
[17] Korennoy Ya A and Man'ko V I 2011 *Phys. Rev.* A **83** 053817
[18] Benichi H and Furusawa A 2011 *Phys. Rev.* A **84** 032104
[19] Filippov S N and Man'ko V I 2011 *Phys. Rev.* A **84** 033827
[20] Ibort A, Man'ko V I, Marmo G, Simoni A and Ventriglia F 2009 *Phys. Scr.* **79** 065013
[21] Filippov S N and Man'ko V I 2011 *Phys. Scr.* **83** 058101
[22] Man'ko V I and Mendes R Vilela 1999 *Phys. Lett.* A **263** 53
[23] Mancini S, Man'ko V I and Tombesi P 1995 *Quantum Semiclass. Opt.* **7** 615
[24] D'Ariano G M, Mancini S, Man'ko V I and Tombesi P 1996 *Quantum Semiclass. Opt.* **8** 1017
[25] Fan H Y and Yang Y L 2006 *Phys. Lett.* A **353** 439
[26] Fan H Y and Guo Q 2006 *Phys. Lett.* A **358** 203
[27] Weyl H 1927 *Z. Phys.* **46** 1
[28] Fan H Y, Zaidi H R and Klauder J R 1987 *Phys. Rev.* D **35** 1831
[29] Wünsche A 2000 *J. Mod. Opt.* **47** 33
[30] Facchi P, Ligabò M and Pascazio S 2010 *J. Mod. Opt.* **57** 239
[31] Wünsche A 1997 *J. Mod. Opt.* **44** 2293
[32] Aksenov V P 1992 *J. Mod. Opt.* **39** 1017
[33] Militello B, Man'ko V I, Man'ko M A and Messina A 2009 *Phys. Scr.* **80** 058102
[34] Fan H Y and Zaidi H R 1987 *Phys. Lett.* A **124** 303
[35] Fan H Y and Guo Q 2007 *Phys. Scr.* **76** 533