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# Relations between scaling-transformed Husimi functions, Wigner functions and symplectic tomograms describing corresponding physical states

**V A Andreev[1], D M Davidović[2], L D Davidović[3] and M D Davidović [4]**

[1] P N Lebedev Physical Institute, Leninskii Prospect 53, Moscow 119991, Russia
[2] Institute for Nuclear Sciences Vinća, 11001 Beograd, Serbia
[3] Institute of Physics, 11080 Beograd, Serbia
[4] Faculty of Civil Engineering, University of Belgrade, 11070 Beograd, Serbia

E-mail: andrvlad@yandex.ru, davidd@vinca.rs, Ljubica.davidovich@gmail.com and davidovich.milena@gmail.com

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**Abstract**

Husimi $Q$-functions are the only functions from the class of Cohen quasi-distributions on phase space that after scaling transformation $(q, p) \to (\lambda q, \lambda p)$ remain in the same class when the modulus of the scaling parameter is smaller than unity and so, in this case, describe a physical state. We found the Wigner functions and symplectic tomograms of such states. We applied the obtained general results to the Fock states of the harmonic oscillator.

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## 1. Introduction

Quantum mechanics in an inherently statistical theory and in it, in general, only the probabilities of experimental outcomes can be predicted. In this respect, it is similar to classical statistical mechanics. The main object of investigation in the latter is the distribution function $\rho(q, p)$. This quantity represents the probability density that the system considered is in the state that is characterized by the parameters $(q, p)$. With the help of it various average values can be calculated. It is defined on the phase space and its arguments represent the coordinate and the momentum of the physical system, and it is also assumed that they can be measured simultaneously with any required accuracy.

As quantum mechanics is also a statistical theory, from the very beginning of its history, attempts have been made to describe quantum states and supply a representation of quantum mechanics that in some sense would be similar to the formulation of classical statistical mechanics.

The oldest phase space formulation of quantum mechanics was introduced by Wigner in 1932. In it every quantum mechanical state was represented by the corresponding function in phase space—its Wigner function $W(q, p)$ [1]. Although it has exact marginal distributions and, in calculations of quantum mechanical average values, plays a role analogous to that of the classical distribution function, it cannot be interpreted as a probability distribution because in the general case it necessarily assumes negative values. Due to this it is a quasi-distribution. Its main features have been described in the review articles [2, 3].

The first positive function on phase space to be introduced, which also completely describes the quantum state, was the Husimi–Kano $Q$-function [4, 5] (following accepted practice, hereafter we call it simply the Husimi function). Also well known is the Glauber–Sudarshan $P$-function [6, 7].

In the papers [8], quantum tomography was introduced. Namely, the function $w(X, \mu, \nu)$ was introduced, which is in one-to-one correspondence with a Wigner function. It may be interpreted as a quantum mechanical probability density of the quantity $X = \mu q + \nu p$. The quantity $X$ may be interpreted as a coordinate in a rotated system of coordinates in phase space $(Q, P)$. The function $w(X, \mu, \nu)$ supplies a complete quantum mechanical description of a quantum state and, furthermore, is a real quantum mechanical probability distribution. It is known as a symplectic tomogram and is described in great detail in [9].

 

In this paper, we analyze various relations, differential and integral, between Wigner functions, Husimi functions and symplectic tomograms. We also analyze the behavior of the Husimi functions under scaling transformation

$$(q, p) \to (\lambda q, \lambda p). \tag{1}$$

After such a transformation and renormalization the Husimi $Q(q, p)$ function becomes $\lambda^2 Q(\lambda q, \lambda p)$. In [10] the proof is given that the so obtained function is again a Husimi function of some physical state if $\lambda$ is a real parameter and $|\lambda| \leqslant 1$. The above transformation arises in a natural way in a number of physical problems and especially in the problem of the most quiet phase-insensitive amplification of a quantum state [11]. In this case, the parameter $\lambda$ is equal to the inverse value of the coefficient of amplification $G = \lambda^{-1}$. In this way, the scaling transformation of Husimi functions is not only a mathematical procedure but also has a clear physical meaning, as an amplification of a quantum state. As Husimi functions in the class of Cohen functions are the only functions that after scaling remain physical states, it is interesting to find how the Wigner functions and tomograms behave in such an amplification of a quantum state. We also analyze this problem for the general case and apply the obtained results to the Fock states.

## 2. Relations between Husimi functions of quantum states and the corresponding Wigner functions and symplectic tomograms

The Husimi function of the quantum state described by a density operator $\hat{\rho}$ is defined as the average value of this operator in the coherent state basis $\langle x|\alpha\rangle$, so that we can write [4, 5]

$$Q(\alpha, \alpha^*) = \frac{1}{\pi} \int \langle \alpha|x\rangle \rho(x, y) \langle y|\alpha\rangle \, \mathrm{d}x \, \mathrm{d}y. \tag{2}$$

Here $\alpha = \alpha_r + \mathrm{i}\alpha_i$ is a complex number, $\rho(x, y)$ is a density matrix in the coordinate representation, while a coherent state in the same basis $\langle x|\alpha\rangle$ may be written in the form

$$n\langle x|\alpha\rangle = \left(\frac{1}{\pi}\right)^{1/4} \exp\left[-\frac{1}{2}(x - \sqrt{2}\alpha)^2 + \mathrm{i}\alpha\alpha_i\right], \tag{3}$$

where $\alpha = (q + \mathrm{i}p)/\sqrt{2}$ and $\hbar = 1$.

In this paper, we analyze the behavior of Husimi functions under scaling transformations and the relations of the scaled functions with Wigner functions and symplectic tomograms of corresponding quantum states.

The Wigner function $W(q, p)$ may be expressed through the density matrix $\hat{\rho}$ in the form

$$W(q, p) = \int \rho\left(q + \frac{u}{2}, q - \frac{u}{2}\right) \mathrm{e}^{-\mathrm{i}pu} \, \mathrm{d}u. \tag{4}$$

Here $\rho(x, x')$ is a density matrix in coordinate representation.

The Husimi function is related to the Wigner function $W(q, p)$ through the following differential relation [12, 13]:

$$Q(q, p) = \exp\left(\frac{1}{4}\frac{\partial^2}{\partial q^2} + \frac{1}{4}\frac{\partial^2}{\partial p^2}\right) W(q, p). \tag{5}$$

The expression (5) may be inverted so that one can express the Wigner function through a Husimi function [10] in the following way:

$$W(q, p) = \exp\left(-\frac{1}{4}\frac{\partial^2}{\partial q^2} - \frac{1}{4}\frac{\partial^2}{\partial p^2}\right) Q(q, p). \tag{6}$$

Using Fourier transforms one can obtain the integral analogues of the differential relations (5) [2, 10]:

$$Q(q, p) = \frac{1}{\pi} \int \exp\left(-(q - q')^2 - (p - p')^2\right) W(q', p') \, \mathrm{d}q' \, \mathrm{d}p'. \tag{7}$$

In [14] the density matrix is expressed through the Husimi function in the form

$$\rho(x, x') = \exp\left[\frac{1}{2}(x - x')^2\right] \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \int \mathrm{d}q \, \mathrm{d}p H_{2n} \times \left[\left(q - \frac{1}{2}(x + x')^2\right)\right] K(q, p; x, x') Q(q, p). \tag{8}$$

Here

$$K(q, p; x, x') = \frac{1}{\sqrt{\pi}} \exp\left[-\left(q - \frac{1}{2}(x + x')\right)^2 - (x - x')^2 + \mathrm{i}p(x - x')\right]. \tag{9}$$

A simpler expression for the density matrix was given in [10] as

$$\begin{aligned}\rho(x, y) &= \frac{1}{2}\sqrt{\pi} \exp\left[\frac{1}{2}(x^2 + y^2)\right] \int Q\left(\frac{\mathrm{i}}{2}(p_1 - p_2), \frac{1}{2}(p_1 + p_2)\right) \\ &\quad \times \exp\left(-\frac{1}{4}(p_1 - p_2)^2 - \mathrm{i}p_1 x + \mathrm{i}p_2 y\right) \mathrm{d}p_1 \, \mathrm{d}p_2.\end{aligned} \tag{10}$$

The inverse relation, namely the expression for the Husimi function through the density matrix, may be obtained using the formulae (5) and (7) and the definition of the Wigner function (4).

The symplectic tomograms were introduced in the papers [8]. They can be represented with the help of the Wigner function ($\hbar = 1$):

$$w(X, \mu, \nu) = \frac{1}{(2\pi)^2} \int \mathrm{d}k \, \mathrm{d}q \, \mathrm{d}p \, W(q, p) \, \mathrm{e}^{-\mathrm{i}k(X - \mu q - \nu p)}. \tag{11}$$

The inverse transformations read

$$W(q, p) = \frac{1}{2\pi} \int \mathrm{d}X \, \mathrm{d}\mu \, \mathrm{d}\nu \, w(X, \mu, \nu) \exp(-\mathrm{i}(\mu q + \nu p - X)). \tag{12}$$

Symplectic tomograms may be also defined through the density matrix of a state in the form $\hat{\rho}$ [15]

$$w(X, \mu, \nu) = \langle X, \mu, \nu|\hat{\rho}|X, \mu, \nu\rangle. \tag{13}$$

Here $|X, \mu, \nu\rangle$ are the eigenstates of the Hermitian operator $\hat{X}_{\mu\nu}$, corresponding to the eigenvalue $X$, while $\mu$ and $\nu$ are real parameters.

On the other hand, we have the formula (7), expressing the Husimi function through the Wigner function. Inserting the expression (12) into the formula (7), we obtain the expression for the Husimi function through a tomogram

$$Q(q,p) = \frac{1}{2\pi}\int \mathrm{d}X\,\mathrm{d}\mu\,\mathrm{d}\nu\, w(X,\mu,\nu)\mathrm{e}^{-(1/4)\mu^2-(1/4)\nu^2} \times \exp(-\mathrm{i}(\mu q+\nu p - X)). \tag{14}$$

Using the expression (6) for the Wigner function and (11), one can express the tomogram as

$$\begin{aligned} w(X,\mu,\nu) &= \frac{1}{(2\pi)^2}\int \mathrm{d}k\,\mathrm{d}q\,\mathrm{d}p\,\exp\left(-\frac{1}{4}\frac{\partial^2}{\partial q^2}-\frac{1}{4}\frac{\partial^2}{\partial p^2}\right) \\ &\quad\times Q(q,p)\,\mathrm{e}^{-\mathrm{i}k(X-\mu q-\nu p)} \\ &= \frac{1}{(2\pi)^2}\int \mathrm{d}k\,\mathrm{d}q\,\mathrm{d}p\,\mathrm{e}^{k^2[(1/4)\mu^2+(1/4)\nu^2]} \\ &\quad\times Q(q,p)\,\mathrm{e}^{-\mathrm{i}k(X-\mu q-\nu p)}. \end{aligned} \tag{15}$$

## 3. Scaling-transformed Husimi functions and the corresponding Wigner functions and tomograms

Let us consider the following transformation of the Husimi function:

$$Q_\lambda(q,p) = \lambda^2 Q(\lambda q, \lambda p). \tag{16}$$

As we already mentioned, it arises in a natural way in several physical situations, and what is especially interesting is that, in [11], with the help of this transformation the amplification of a quantum state was described. It was shown there that in the case of the most quiet phase insensitive amplification, the Husimi functions of input and output quantum states are related through the relation

$$Q_{\mathrm{out}}(\alpha) = \frac{1}{G^2}Q_{\mathrm{in}}\left(\frac{\alpha}{G^2}\right), \tag{17}$$

where $G$ is the coefficient of amplification. It was shown in [16] that, if $\lambda^2 \leqslant 1$ and $Q(q,p)$ is a Husimi function, then the function $Q_\lambda(q,p)$ is also a Husimi function of some quantum state. It is also well known that the analogous transformation of the Wigner function, namely the transformation of the form

$$\tilde{W}^\lambda(q,p) = \lambda^2 W(\lambda q, \lambda p), \tag{18}$$

is not positive definite and so does not lead from a Wigner function to a Wigner function again. In [16] this was shown for the case of the first excited state of the harmonic oscillator. In (18) this transformation was applied to the Wigner function of the mentioned state.

However, we can construct the Wigner function that as a function describing the physical state corresponds to the Husimi function generated by transformation (1). To this end, as the first step from the considered Wigner function using the formulae (5) and (7) the corresponding Husimi function should be constructed. Then to this Husimi function the transformation (1) should be applied and then for the so obtained Husimi function the corresponding Wigner function using the formula (6) should be found. Such a procedure leads from a Wigner function again to a Wigner function. Let us find its explicit form.

The Wigner function can be expressed through the Husimi function using the relation (6). For the scaled Husimi function (16), this relation takes the form

$$W^\lambda(q,p) = \exp\left(-\frac{1}{4}\frac{\partial^2}{\partial q^2}-\frac{1}{4}\frac{\partial^2}{\partial p^2}\right)\lambda^2 Q(\lambda q,\lambda p). \tag{19}$$

The expression (19) may be represented in the form

$$W^\lambda(q,p) = \lambda^2\exp\left(-\frac{1}{4}\frac{\partial^2}{\partial q^2}-\frac{1}{4}\frac{\partial^2}{\partial p^2}+\frac{1}{4\lambda^2}\frac{\partial^2}{\partial q^2}+\frac{1}{4\lambda^2}\frac{\partial^2}{\partial p^2}\right) \times W(\lambda q,\lambda p). \tag{20}$$

The formula (20) shows the method of transformation of the Wigner function in the case when the corresponding Husimi function is transformed according to rule (16).

The obtained result (20) being in a differential form is an analogue of corresponding relations (5) and (6). For the Wigner functions $W(q,p)$, $W^\lambda(q,p)$ it is also possible to find the integral relation that connects them, which is analogous to equation (7). To this end one can use the Fourier transform of the function $W(\lambda q, \lambda p)$.

$$W(u,v) = \int \mathrm{d}q'\,\mathrm{d}p'\,\mathrm{e}^{\mathrm{i}uq'+\mathrm{i}vp'}W(\lambda q',\lambda p'). \tag{21}$$

Inverting this Fourier transform, one obtains the following identity:

$$\begin{aligned} &W(\lambda q,\lambda p) \\ &\quad= \frac{1}{2\pi}\int \mathrm{d}q''\,\mathrm{d}p''\,\mathrm{d}q'\,\mathrm{d}p'\,\mathrm{e}^{\mathrm{i}(q'-q)p''+\mathrm{i}(p'-p)q''}W(\lambda q',\lambda p'). \end{aligned} \tag{22}$$

Now, to the obtained expression (22) the following operator should be applied:

$$\lambda^2\exp\left(-\frac{1}{4}\left(1-\frac{1}{\lambda^2}\right)\frac{\partial^2}{\partial q^2}-\frac{1}{4}\left(1-\frac{1}{\lambda^2}\right)\frac{\partial^2}{\partial p^2}\right).$$

Using the formula (22), we obtain

$$\begin{aligned} W^\lambda(q,p) &= \frac{\lambda^2}{(2\pi)^2}\int \mathrm{d}q''\,\mathrm{d}p''\,\mathrm{d}q'\,\mathrm{d}p'\,\exp\left(\frac{1}{4}\left(1-\frac{1}{\lambda^2}\right)(p'')^2\right. \\ &\left.+\frac{1}{4}\left(1-\frac{1}{\lambda^2}\right)(q'')^2\right)\mathrm{e}^{\mathrm{i}(q-q')p''+\mathrm{i}(p-p')q''}W(\lambda q',\lambda p'). \end{aligned} \tag{23}$$

The final result is

$$\begin{aligned} W^\lambda(q,p) &= \frac{1}{\pi}\frac{\lambda^2}{1-\lambda^2}\int \mathrm{d}q''\,\mathrm{d}p'' \\ &\quad\times\exp\left[-\frac{(\lambda q-q'')^2}{(1-\lambda^2)}-\frac{(\lambda p-p'')^2}{(1-\lambda^2)}\right]W(q'',p''). \end{aligned} \tag{24}$$

The expression (24) gives the integral relation between the Wigner function $W(q,p)$, which corresponds to the Husimi function $Q(q,p)$, and the Wigner function

$W^\lambda(q,p)$, which corresponds to the scaled Husimi function $Q_\lambda(q,p)=\lambda^2 Q(\lambda q,\lambda p)$.

Let us explain now what happens to the initial tomogram when a Husimi function is scale transformed according to (16).

As we already found, the tomogram may be expressed through the Husimi function using the formula (15). Assuming that in this formula already transformed functions are present, we will write it in the form

$$w^\lambda(X,\mu,\nu)=\frac{1}{(2\pi)^2}\int \mathrm{d}k\,\mathrm{d}q\,\mathrm{d}p\,\mathrm{e}^{k^2[(1/4)\mu^2+(1/4)\nu^2]}\times Q_\lambda(q,p)\,\mathrm{e}^{-\mathrm{i}k(X-\mu q-\nu p)}. \tag{25}$$

Inserting into the formula (25) the expression (16) for the transformed Husimi function and using the formula (14), which relates the Husimi function through the tomogram, the following expression relating the transformed tomogram $w^\lambda$ with the initial one $w$ may be obtained:

$$w^\lambda(X,\mu,\nu)=\frac{\lambda^2}{2\pi}\int \mathrm{d}Y\,\mathrm{d}k\,w\left(Y,\frac{k\mu}{\lambda},\frac{k\nu}{\lambda}\right)\times\exp\left[\frac{k^2}{4}\mu^2\left(1-\frac{1}{\lambda^2}\right)+\frac{k^2}{4}\mu^2\left(1-\frac{1}{\lambda^2}\right)\right]\mathrm{e}^{\mathrm{i}(Y-kX)}. \tag{26}$$

## 4. The harmonic oscillator

We will now, as an example, apply the general obtained results to the case of the harmonic oscillator. We wish to find how the Fock state of the harmonic oscillator is transformed when the Husimi function of the initial state is scale transformed according to (16). To this end, let us consider the following concrete operator:

$$\hat\rho_N=\frac{\lambda^{2N+2}}{N!}\sum_{k=0}^{\infty}\frac{(N+k)!}{k!}(1-\lambda^2)^k|N+k\rangle\langle N+k|,\qquad \lambda^2<1. \tag{27}$$

The operator (27) may evidently be interpreted as the density matrix of some quantum state. From the form of the expression (27), it is evident that the described state is a mixed state that consists of a linear combination of the pure states $|N+k\rangle$, $k=0,1,2,\ldots,\infty$. Each of these pure states $|N+k\rangle$ is present in the mixed state with the probability

$$c_k^N=\frac{\lambda^{2N+2}(N+k)!}{N!k!}(1-\lambda^2)^k. \tag{28}$$

Let us find now the Husimi function of the state (27):

$$Q_{\rho_N}(q,p)=\langle\alpha|\hat\rho_N|\alpha\rangle=\frac{\lambda^{2N+2}}{N!}\sum_{k=0}^{\infty}\frac{(N+k)!}{k!}(1-\lambda^2)^k\frac{|\alpha|^{2(N+k)}}{(N+k)!}\mathrm{e}^{-|\alpha|^2}=\frac{\lambda^{2N+2}}{N!}|\alpha|^{2N}\mathrm{e}^{-|\alpha|^2}\sum_{k=0}^{\infty}\frac{(1-\lambda^2)^k}{k!}|\alpha|^{2k}=\frac{\lambda^2}{N!}\lambda^{2N}|\alpha|^{2N}\,\mathrm{e}^{-\lambda^2|\alpha|^2}. \tag{29}$$

Now, the Husimi function of the pure Fock state $|N\rangle$ is given by

$$Q_N(q,p)=\langle\alpha|N\rangle\langle N|\alpha\rangle=\frac{1}{N!}|\alpha|^{2N}\,\mathrm{e}^{-|\alpha|^2}. \tag{30}$$

Applying to the Husimi function $Q_N$ the scaling transformation (16), we obtain

$$Q_N^\lambda(q,p)=\lambda^2 Q_N(\lambda q,\lambda p)=\frac{\lambda^2}{N!}\lambda^{2N}|\alpha|^{2N}\,\mathrm{e}^{-\lambda^2|\alpha|^2}. \tag{31}$$

Comparing the expressions (29) and (31), we can see that they are equal. From this fact it follows that after scaling transformation of the corresponding Husimi functions (16) the pure Fock state $|N\rangle$ becomes the mixed state, given by the density matrix $\hat\rho_N$.

## 5. Conclusion

In this paper, we have analyzed the relations between Husimi functions, Wigner functions and simplectic tomograms. We found explicit differential and integral expressions for these relations. We also found in an explicit form the density matrix for the scaling transformed Husimi functions of Fock states for the harmonic oscillator.

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