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Energy-sensitive and "Classical-like" Distances between Quantum States

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Abstract

We introduce the concept of the "polarized" distance, which distinguishes the orthogonal states with different energies. We also give new inequalities for the known Hilbert-Schmidt distance between neighbouring states and express this distance in terms of the quasiprobability distributions and the normally ordered moments. Besides, we discuss the distance problem in the framework of the recently proposed "classical-like" formulation of quantum mechanics, based on the sympletic tomography scheme. The examples of Fock's, coherent, "Schrödinger cats", squeezed, phase and thermal states are considered.

1. Introduction

Last year, an increasing interest to the problem of distance between quantum states is observed. Different motivations of this activity can be found in such fields as quantum cryptography, quantum communications, or quantum computing. Here we discuss the topic mainly from the point of view of quantum optics. In view of recent impressive progress in creating and detecting various types of nonclassical states of light or cooled particles in electromagnetic traps, the problem of measures of distinguishability or closeness between different quantum states becomes actual. For example, in quantum optics, Glauber's coherent states [1]

$$|\alpha\rangle = \exp\left(-|\alpha|^2/2\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{1.1}$$

are considered frequently as reference states ($|n\rangle$ means the Fock state with the definite number of photons), so that the (pure) states different from (1.1) are called sometimes as *nonclassical states*. But what is the quantitative measure of the "nonclassicality"? The simplest option is to use the so called Mandel's parameter, $\mathcal{Q} = \overline{n^2}/\bar{n} - \bar{n} - 1$, which equals zero for all coherent states, since they have the Poissonian photon statistics. However, this parameter is adequate for a limited class of states. Consider, for instance, the *even and odd coherent states* introduced in [2]

$$|\alpha;\pm\rangle = (2[1 \pm \exp(-2|\alpha|^2)])^{-1/2}(|\alpha\rangle \pm |-\alpha\rangle). \tag{1.2}$$

In this case Mandel's parameter equals $\mathcal{Q}^{(\pm)} = \pm 2|\alpha|^2/\sinh(2|\alpha|^2)$, and it shows distinctly the qualitative difference between the states $|\alpha\rangle$, $|\alpha;+\rangle$, and $|\alpha;-\rangle$, but only for small values of $|\alpha|$. If $|\alpha| \gg 1$, then $\mathcal{Q}^{(\pm)} \approx 0$, although the states $|\alpha;\pm\rangle$ are still quite different from the coherent state. Moreover, for *generalized coherent states* [3, 4]

$$|\tilde{\alpha}\rangle = \exp\left(-|\alpha|^2/2\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \exp\left[i\varphi(n)\right] |n\rangle \tag{1.3}$$

we have identically $\mathcal{Q} \equiv 0$ for any function $\varphi(n)$, although the state $|\tilde{\alpha}\rangle$ may be essentially different from the Glauber state $|\alpha\rangle$. For example, the choice $\varphi(2k) = 0 (\mathrm{mod}\ 2\pi)$, $\varphi(2k+1) = -\pi/2 (\mathrm{mod}\ 2\pi)$ gives the so-called Yurke–Stoler state [5]

$$|\tilde{\alpha}\rangle_{\mathrm{YS}} = e^{-i\pi/4}(|\alpha\rangle + i|-\alpha\rangle)/\sqrt{2} \tag{1.4}$$

which is considered, equally with the even and odd states, as a representative of a large family of "Schrödinger cat states".

The concept of distance gives a possibility to characterize more precisely the neighbourhood or similarity between the quantum states. However, the existing approaches (see Section 2) seem to suffer from certain drawbacks. Some of the available definitions of a distance are too complicated to perform concrete calculations. On the other hand, some consequences of the traditional approaches, being correct mathematically, contradict the physical intuition. For example, the known definitions yield the same, at once, maximum possible value of the distance between any two orthogonal pure states, whereas from the physical point of view, the distance between the first and the 100th Fock states seems to be much greater than that between, say, the 100th and the 101st states. The distance measures based on the density operators alone are not sensitive to the difference in energies.

In the present paper we propose new measures which distinguish different orthogonal states and which are simple enough to perform the calculations, at least for the most important families of states used in quantum optics. In our approach, the distance depends not only on the density operators alone, but also on some extra fixed positively definite operator. Of course, following this way we meet the problem of the nonuniqueness in the choice of this additional "polarization" operator. Nonetheless, such a uniqueness seems not crucial in many physical applications, where the special role of some operators (like the Hamiltonian or the quantum number operator) is evident from the beginning. Another goal is to provide an analysis of the distance problem in terms of the quasiprobability distributions and in the framework of the "classical-like" formulation of quantum mechanics proposed recently in [6].

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The paper is organized as follows. In Section 2 we give a review of the existing approaches to the quantum distance problem. In Section 3 we concentrate on the properties of the Hilbert–Schmidt distance (HSD) and we express it in terms of the quasiprobability functions and ordered moments. In Section 4 we propose several definitions of the energy-sensitive distance in terms of the statistical operators (density matrices). In Section 5 the distinctions between different definitions are illustrated by examples of the Fock, coherent, "Schrödinger cat", squeezed, phase and thermal states. The "classical-like" distances between quantum states are considered in Section 6. The last section contains brief conclusions.

2. Previous approaches to the quantum distance problem

The distance between two objects a and b is defined usually as a scalar real function satisfying the following properties:

$$\text{(I)} \quad d(a, a) = 0,\ d(a, b) > 0, \quad \text{if } a \neq b, \tag{2.1}$$

$$\text{(II)} \quad d(a, b) = d(b, a), \tag{2.2}$$

$$\text{(III)} \quad d(a, b) + d(b, c) \geqslant d(a, c). \tag{2.3}$$

The property (III) has a clear geometrical meaning as the *triangle inequality*, and it implies rather strong limitations on the possible choice of the function $d(a, b)$. If the "objects" a and b are different pure quantum states, then the distance must be some functional written in terms of the Hilbert space vectors, $|a\rangle$ and $|b\rangle$, representing the states. One should remember, however, that the set of quantum states is in one-to-one correspondence not with the whole Hilbert space of the wave functions, but with its projective factor space, since the vectors $|\psi\rangle$ and $e^{i\varphi}|\psi\rangle$ describe the same state. All the requirements are satisfied, e.g. for the *Fubiny-Study distance* [7, 8, 9]

$$d^{(FS)}(\psi_1, \psi_2) = \sqrt{2}(1 - |\langle\psi_1|\psi_2\rangle|^2)^{1/2} \tag{2.4}$$

(sometimes the factor $\sqrt{2}$ is replaced by 1 or 2), although a slightly different definition

$$d^{(\min)}(\psi_1, \psi_2) = \inf_{\varphi} \| |\psi_1\rangle - e^{i\varphi}|\psi_2\rangle \| = \sqrt{2}(1 - |\langle\psi_1|\psi_2\rangle|)^{1/2} \tag{2.5}$$

is also possible [10]. Taking a one-parameter family of states $\psi(t)$ generated by the time evolution operator, one obtains, both from (2.4) and (2.5), the infinitesimal distance along the evolution curve in the projective Hilbert space

$$ds = \sqrt{2 - 2|\langle\psi(t)|\psi(t + dt)\rangle|^2} \approx 2\sqrt{1 - |\langle\psi(t)|\psi(t + dt)\rangle|}\,. \tag{2.6}$$

The definition (2.6) was used in studies devoted to the geometrical aspects of the quantum evolution and generalizations of the time-energy uncertainty relations [9, 10, 11, 12, 13, 14, 15, 16, 17]. For a family of states $\psi(s)$ dependent on a continuous vector parameter $s = (s_1, s_2, \ldots, s_n) \in \mathbf{R}^n$ one can introduce the Riemannian metrics according to $\|\psi(s + ds) - \psi(s)\|^2 = \gamma_{ij}\, ds_i\, ds_j$ and measure not the "shortest" distance (3.1), but the distance along a geodesics on a curved manifold, which can be much greater than the "shortest" one. The concrete examples of the geometries on the manifolds corresponding to the most known continuous families of quantum states (namely, coherent, squeezed and displaced states) were studied in detail in [16, 18, 19, 20, 21, 22].

Wootters [23] proposed the distance between the pure states in the form of the angle between the corresponding rays in the Hilbert space $d^{(W)}(|\psi_1\rangle, |\psi_2\rangle) = \cos^{-1}|\langle\psi_2|\psi_1\rangle|$. For infinitesimally close states, the differential form of this distance coincides (up to a coefficient) with (2.6) [24]. Recently, the Wootters and Fubini-Study metrics were compared in [25].

Now let us turn to the *mixed* quantum states, described by positively definite statistical operators $\hat{\rho}$ with the unit trace: $\mathrm{Tr}\hat{\rho} = 1$. The first definition of the distance between mixed states in the physical literature, perhaps, was given in [26]

$$d^{(JMG)}(\hat{\rho}_1, \hat{\rho}_2) = \sup_{\|A\| = 1} |\mathrm{Tr}([\hat{\rho}_1 - \hat{\rho}_2]\hat{A})|\,. \tag{2.7}$$

Restricting the family of the bounded operators $\hat{A}$ in this definition by the projection operators $\hat{E} = \hat{E}^2$, one obtains an equivalent definition [27]

$$d^{(JMG)}(\hat{\rho}_1, \hat{\rho}_2) = \sup_{E} |\mathrm{Tr}([\hat{\rho}_1 - \hat{\rho}_2]\hat{E})| = \tfrac{1}{2}\|\hat{\rho}_1 - \hat{\rho}_2\|_1, \tag{2.8}$$

where $\|\hat{A}\|_1 \equiv \mathrm{Tr}\sqrt{\hat{A}^\dagger\hat{A}} \equiv \sum|\lambda_n|$, the summation being performed over all the eigenvalues λ_n of the operator $\hat{A}$. Actually, the right-hand side of eq. (2.8) was used by Hillery [28] as a starting point in his definition of the distance between a state $\hat{\rho}$ and a given family of "classical" states $\hat{\rho}_{cl}$ as $\delta = \inf_{\rho_{cl}}\|\hat{\rho} - \hat{\rho}_{cl}\|_1$. More sophisticated definitions of the distance were given, e.g. in [29, 30]. However, they are so complicated from the point of view of calculations, that no explicit examples were considered.

One of the most frequently cited in the physical literature definitions is the so-called *Bures–Uhlmann distance* (BU-distance) [31, 32]. It has the form (see also [27, 33, 34, 35])

$$d^{(BU)}(\hat{\rho}_1, \hat{\rho}_2) = (2 - 2\,\mathrm{Tr}\sqrt{\hat{\rho}_1^{1/2}\hat{\rho}_2\,\hat{\rho}_1^{1/2}})^{1/2} \tag{2.9}$$

where the operator $\hat{\rho}^{1/2}$ is defined as the *positively semi-definite Hermitian operator* satisfying the relation $(\hat{\rho}^{1/2})^2 = \hat{\rho}$. This operator is unique. Although the right-hand side of (2.9) seems asymmetrical with respect to $\hat{\rho}_1$ and $\hat{\rho}_2$, actually $d^{(BU)}(\hat{\rho}_1, \hat{\rho}_2) = d^{(BU)}(\hat{\rho}_2, \hat{\rho}_1)$ [35]. For *pure* quantum states $\hat{\rho}_\psi = |\psi\rangle\langle\psi|$ the BU-distance coincides with the "minimal" distance (2.5) due to the relations $\hat{\rho}_\psi^{1/2} = \hat{\rho}_\psi^2 = \hat{\rho}_\psi$. If one of the states is pure, then

$$d^{(BU)}(|\psi\rangle\langle\psi|, \hat{\rho}) = \sqrt{2}(1 - \sqrt{\langle\psi|\hat{\rho}|\psi\rangle})^{1/2}. \tag{2.10}$$

However, the calculations are much more involved in the generic case of nondiagonal statistical operators, so that the explicit forms of the Bures–Uhlmann distance were found only for finite-dimensional $N \times N$ density matrices (especially for $N = 2$ and $N = 3$) [34, 36, 37] and, recently, for squeezed thermal states [38, 39] and displaced thermal states [40].

3. Distances based on the Hilbert–Schmidt norm

A simple expression for the distance between quantum states, enabling to perform calculations for the most important classes of states (at least in the problems of quantum optics), is based on the Hilbert–Schmidt norm $\|\hat{A}\|_2 \equiv \sqrt{\mathrm{Tr}(\hat{A}^\dagger\hat{A})}$. The Hilbert–Schmidt distance (HSD) of two sta-

tistical operators $\hat{\rho}_1$ and $\hat{\rho}_2$ is defined as [8, 9, 16, 27, 41, 42, 43]

$$d^{(\mathrm{HS})}(\hat{\rho}_1, \hat{\rho}_2) = \|\hat{\rho}_1 - \hat{\rho}_2\|_2 = \{\mathrm{Tr}\,[(\hat{\rho}_1 - \hat{\rho}_2)^2]\}^{1/2}$$
$$= [\mathrm{Tr}\,(\hat{\rho}_1^2) + \mathrm{Tr}\,(\hat{\rho}_2^2) - 2\,\mathrm{Tr}\,(\hat{\rho}_1\hat{\rho}_2)]^{1/2}. \tag{3.1}$$

In particular (we write simply d instead of $d^{(\mathrm{HS})}$ in all cases when it does not lead to a confusion),

$$d(|\psi\rangle\langle\psi|, \hat{\rho}) = [1 + \mathrm{Tr}\,(\hat{\rho}^2) - 2\langle\psi|\hat{\rho}|\psi\rangle]^{1/2}$$
$$\leqslant \sqrt{2}\,[1 - \langle\psi|\hat{\rho}|\psi\rangle]^{1/2}, \tag{3.2}$$

so the HSD (3.1) goes to the Fubini–Study distance (2.4) in the special case of two pure states. The possible values of the Hilbert–Schmidt distance are restricted by $0 \leqslant d(\hat{\rho}_1, \hat{\rho}_2) \leqslant \sqrt{2}$, the maximum $\sqrt{2}$ being reached for any pair of orthogonal (pure) states.

In many cases it is convenient to describe the quantum states with the aid of quasiprobability distributions, which can be written as special cases of the general Cahill–Glauber s-distribution [44]

$$W(\alpha, s) = \mathrm{Tr}\,[\hat{\rho}\hat{T}(\alpha, s)], \tag{3.3}$$

where

$$\hat{T}(\alpha, s) = \int \frac{\mathrm{d}^2\zeta}{\pi} \exp\left[\zeta(\hat{a}\dagger - \alpha^*) - \zeta^*(\hat{a} - \alpha) + \frac{s}{2}|\zeta|^2\right],$$

α, ζ are complex numbers and $\hat{a}$, $\hat{a}\dagger$ are the boson annihilation and creation operators (in one dimension for simplicity). The choice $s = 0$ (with $\alpha = (q + \mathrm{i}p)/\sqrt{2}$) yields the Wigner function [45] $W(q, p) \equiv \int \mathrm{d}u \exp(\mathrm{i}pu)\langle q - u/2|\hat{\rho}|q + u/2\rangle$. For $s = -1$ we have the so called Husimi–Kano or Q-function [46] $W(\alpha, -1) \equiv Q(\alpha) = \langle\alpha|\hat{\rho}|\alpha\rangle$, whereas in the case $s = +1$ we arrive at the Glauber–Sudarshan function $P(\alpha) \equiv W(\alpha, +1)$ which yields the "diagonal" representation of the statistical operator [47] $\hat{\rho} = \int P(\alpha)|\alpha\rangle\langle\alpha|\mathrm{d}^2\alpha/\pi$. Using (3.3) one can write the Hilbert–Schmidt distance in terms of integrals over the phase space:

$$d^2(\hat{\rho}_1, \hat{\rho}_2) = \int \frac{\mathrm{d}q\,\mathrm{d}p}{2\pi}\,[W_1(q, p) - W_2(q, p)]^2 \tag{3.4}$$

$$= \int \frac{\mathrm{d}^2\alpha}{\pi}\,[Q_1(\alpha) - Q_2(\alpha)][P_1(\alpha) - P_2(\alpha)] \tag{3.5}$$

$$= \int \frac{\mathrm{d}^2\alpha\,\mathrm{d}^2\beta}{\pi}\,\frac{\mathrm{d}^2\beta}{\pi}\,\mathrm{e}^{-|\alpha-\beta|^2}$$
$$\times\,[P_1(\alpha) - P_2(\alpha)][P_1(\beta) - P_2(\beta)]. \tag{3.6}$$

If one knows (e.g., from experimental data) all normally ordered moments $M^{(k,l)} = \mathrm{Tr}\,(\hat{a}\dagger^k\hat{a}^l\hat{\rho})$, then the statistical operator $\hat{\rho}$ can be reconstructed as follows [48–50]:

$$\hat{\rho} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} M^{(k,l)}\hat{a}_{k,l}, \quad \hat{a}_{k,l} \equiv \sum_{j=0}^{\min\{k,l\}} \frac{(-1)^j|l-j\rangle\langle k-j|}{j!\sqrt{(k-j)!(l-j)!}}. \tag{3.7}$$

Using this formula one can write the Hilbert–Schmidt distance in the form of a series

$$d^2(\hat{\rho}_1, \hat{\rho}_2) = \sum_{s=0}^{\infty}\sum_{k=0}^{s}\sum_{l=0}^{s} \frac{(-1)^{s+k+l}s!}{k!(s-k)!l!(s-l)!}$$
$$\times\,\Delta M^{(k,l)}\Delta M^{(s-k,s-l)}, \tag{3.8}$$

where $\Delta M^{(k,l)} \equiv M_1^{(k,l)} - M_2^{(k,l)}$. For example, in the case of the coherent state $|\alpha\rangle$ one has $M^{(k,l)} = \alpha^{*k}\alpha^l$ and (3.8) converges to the closed expression (5.1).

An advantage of the Hilbert–Schmidt distance is that it permits to obtain simple inequalities for the distances between neighbouring states. Consider, for example, the distance between an arbitrary state $\hat{\rho}$ and the vacuum state $|0\rangle\langle 0|$. Using formula (3.2) and the identities $\Sigma\langle n|\hat{\rho}|n\rangle \equiv 1$, $\Sigma n\langle n|\hat{\rho}|n\rangle \equiv \bar{n}$, one can write the following chain of relations:

$$d(\hat{\rho}, |0\rangle\langle 0|) \leqslant [2(1 - \langle 0|\hat{\rho}|0\rangle)]^{1/2} = \left[2\sum_{n=1}^{\infty}\langle n|\hat{\rho}|n\rangle\right]^{1/2}$$
$$\leqslant \left[2\sum_{n=1}^{\infty} n\langle n|\hat{\rho}|n\rangle\right]^{1/2} = \sqrt{2\bar{n}}. \tag{3.9}$$

This inequality is useful if $\bar{n} \ll 1$. For an arbitrary reference Fock state $|n\rangle\langle n|$ one can prove in a similar way the inequalities

$$d(\hat{\rho}, |n\rangle\langle n|) \leqslant \sqrt{2}\,[\langle 0|\hat{\rho}|0\rangle + \bar{n} - n\langle n|\hat{\rho}|n\rangle]^{1/2}, \tag{3.10}$$

$$d(\hat{\rho}, |n\rangle\langle n|) \leqslant \sqrt{2}\,[\sigma_n + (n - \bar{n})^2]^{1/2} \tag{3.11}$$

where $\sigma_n \equiv \overline{n^2} - (\bar{n})^2$ is the variance of the number operator in the state $\hat{\rho}$.

In general, one may identify the quantum state not necessarily with the statistical operator $\hat{\rho}$, but with any function of this operator $f(\hat{\rho})$. As a consequence, a whole family of the modified Hilbert–Schmidt distances can be introduced according to the definition

$$\Delta_f(\hat{\rho}_1, \hat{\rho}_2) = \|f(\hat{\rho}_1) - f(\hat{\rho}_2)\|_2$$
$$= (\mathrm{Tr}\{[f(\hat{\rho}_1) - f(\hat{\rho}_2)]^2\})^{1/2}$$
$$= (\mathrm{Tr}\,[f^2(\hat{\rho}_1)] + \mathrm{Tr}\,[f^2(\hat{\rho}_2)]$$
$$- 2\mathrm{Tr}\,[f(\hat{\rho}_1)f(\hat{\rho}_2)])^{1/2}. \tag{3.12}$$

For pure states, Δ_f-distances coincide with the Fubini–Study distance (2.4) for any reasonable function $f(\hat{\rho})$. However, for mixed states the new distances are essentially different. For example, choosing $f(\hat{\rho}) = \hat{\rho}^{1/2}$ we obtain the distance

$$\tilde{d}(\hat{\rho}_1, \hat{\rho}_2) = [2 - 2\mathrm{Tr}\,(\hat{\rho}_1^{1/2}\hat{\rho}_2^{1/2})]^{1/2} \tag{3.13}$$

which coincides with the Bures–Uhlmann distance (2.9) for any *commuting* operators $\hat{\rho}_1$ and $\hat{\rho}_2$ (remember that the pure state projection operators $|\psi\rangle\langle\psi|$ and $|\varphi\rangle\langle\varphi|$ do not commute if $|\psi\rangle \neq |\varphi\rangle$). If one of the states is pure, then

$$\tilde{d}(|\psi\rangle\langle\psi|, \hat{\rho}) = \;= \sqrt{2}\,[1 - \langle\psi|\hat{\rho}|\psi\rangle]^{1/2} \tag{3.14}$$

so the inequalities (3.9)–(3.11) hold for the $\tilde{d}$-distance, as well.

4. Energy-sensitive distance between quantum states

The Hilbert–Schmidt distance between any states cannot exceed the limit value $\sqrt{2}$. In principle, one could "stretch" the distance between remote states, introducing some monotonous function $F(d)$ with the property $F(\sqrt{2}) = \infty$. But such a simple modification yields the same (although infinite) distance for any pair of orthogonal states.

To distinguish orthogonal states with different sets of quantum numbers, we have to break the symmetry of the

Hilbert space with respect to "rotations" of the basis, i.e. to fix some "direction" given by a *positively definite Hermitian* "reference" operator $\hat{Z}$. However, we still want to use the advantage of the Hilbert–Schmidt norm. So, we define the "Z-polarized" distance as

$$d_Z(\hat{\rho}_1, \hat{\rho}_2) = \|\hat{Z}^{1/2}(\hat{\rho}_1 - \hat{\rho}_2)\|_2 = [\mathrm{Tr}\,(\hat{Z}[\hat{\rho}_1 - \hat{\rho}_2)^2]^{1/2}$$
$$= [\mathrm{Tr}\,(\hat{Z}[\hat{\rho}_1^2 + \hat{\rho}_2^2 - \hat{\rho}_1\hat{\rho}_2 - \hat{\rho}_2\hat{\rho}_1])]^{1/2}. \quad (4.1)$$

Another possible definition is

$$\tilde{d}_Z(\hat{\rho}_1, \hat{\rho}_2) = \|\hat{Z}^{1/2}(\hat{\rho}_1^{1/2} - \hat{\rho}_2^{1/2})\|_2$$
$$= [\mathrm{Tr}\,(\hat{Z}[\hat{\rho}_1^{1/2} - \hat{\rho}_2^{1/2}]^2)]^{1/2}$$
$$= [\mathrm{Tr}\,(\hat{Z}[\hat{\rho}_1 + \hat{\rho}_2 - \hat{\rho}_1^{1/2}\hat{\rho}_2^{1/2} - \hat{\rho}_2^{1/2}\hat{\rho}_1^{1/2}])]^{1/2}. \quad (4.2)$$

Evidently, both the definitions satisfy all the axioms due to the properties of the Hilbert–Schmidt norm (since we simply apply this norm to the "scaled" operators $\hat{Z}^{1/2}\hat{\rho}$ or $\hat{Z}^{1/2}\hat{\rho}^{1/2}$). In the special case of pure quantum states $\hat{\rho}_i = |\psi_i\rangle\langle\psi_i|$ we have

$$d_Z^2(|\psi_1\rangle, |\psi_2\rangle) = \tilde{d}_Z^2(|\psi_1\rangle, |\psi_2\rangle)$$
$$= \langle\psi_1|\hat{Z}|\psi_1\rangle + \langle\psi_2|\hat{Z}|\psi_2\rangle$$
$$- \langle\psi_1|\hat{Z}|\psi_2\rangle\langle\psi_2|\psi_1\rangle$$
$$- \langle\psi_2|\hat{Z}|\psi_1\rangle\langle\psi_1|\psi_2\rangle. \quad (4.3)$$

If $\hat{Z}$ coincides with the unity operator, (4.3) goes to the Fubini–Study distance (2.4).

A possibility of using some extra operators to define the distance was mentioned in study [26] whose authors considered the construction $\mathrm{Tr}\,(\hat{A}[\hat{\rho}_1 - \hat{\rho}_2])$. However, it was rejected on the grounds of the unboundedness, if *all* observables A are admitted (the authors of [26] started from the rough definition: "Two states are close to each other if all the expectation values of observables are close to each other"). Here we *fix* the operator $\hat{Z}$, depending on the concrete physical problem.

In the case of quantum optics, a natural choice of $\hat{Z}$ is the quantum number operator

$$\hat{N} = \hat{a}^\dagger\hat{a}. \quad (4.4)$$

Then the N-distance between the Fock states $|n\rangle$ and $|m\rangle$ reads

$$d_N(|m\rangle, |n\rangle) = (1 - \delta_{mn})\sqrt{m+n}. \quad (4.5)$$

We see that $d_N(|m\rangle, |0\rangle) > d_N(|n\rangle, |0\rangle)$ if $m > n$, i.e. higher the energy, more is the distance from the ground state. Nonetheless, the N-distance also does not seem to be ideal. Consider, for instance, two Fock states with $m, n \gg 1$. Then $d_N(|m\rangle, |n\rangle) \gg 1$, even if $|m - n| \sim 1$. Such a property of the distance (4.1) does not agree completely with our intuition. This drawback can be removed, if we assume the following definition:

$$D_Z^2(\hat{\rho}_1, \hat{\rho}_2) = \mathrm{Tr}\,(\Delta\hat{\rho}\hat{Z}\Delta\hat{\rho}) - \frac{[\mathrm{Tr}\,(\Delta\hat{\rho}\hat{Z}^{1/2}\Delta\hat{\rho})]^2}{\mathrm{Tr}\,(\Delta\hat{\rho})^2} \quad (4.6)$$

where $\Delta\hat{\rho} \equiv \hat{\rho}_1 - \hat{\rho}_2$. The right-hand side of eq. (4.6) is non-negative, since it can be written as

$$D_Z^2 = \mathrm{Tr}\,(\Delta\hat{\rho})^2\langle(Z^{1/2} - \langle Z^{1/2}\rangle)^2\rangle \quad (4.7)$$

where the average value is defined as $\langle Z\rangle \equiv \mathrm{Tr}\,(\Delta\hat{\rho}\hat{Z}\Delta\hat{\rho})/\mathrm{Tr}\,(\Delta\hat{\rho})^2$. We shall cautiously name D_Z as a *quasidistance*, since we have no proof of the triangle inequality for *any* states. Applying (4.6) with $\hat{Z} = \hat{N}$ to the Fock's states, we obtain

$$D_N(|n\rangle, |m\rangle) = |\sqrt{n} - \sqrt{m}|/\sqrt{2}. \quad (4.8)$$

This expression obviously satisfies the triangle inequality. Moreover, it is in agreement with the representation of the Fock states in the phase space as circles whose radii are proportional to the square root of the energy [51, 52]. In such a case the distance between the 100th and 101st states is less than that between the ground and the first excited states.

A disadvantage of the definition (4.6) is that it complicates significantly calculations for non-Fock states. In the case of coherent states the calculations are simplified if one slightly modifies the definition of the quasidistance in the following way:

$$\tilde{D}_a^2(\hat{\rho}_1, \hat{\rho}_2) = \mathrm{Tr}\,(\Delta\hat{\rho}\hat{a}\dagger\hat{a}\Delta\hat{\rho}) - \frac{|\mathrm{Tr}\,(\Delta\hat{\rho}\hat{a}\Delta\hat{\rho})|^2}{\mathrm{Tr}\,(\Delta\hat{\rho})^2}. \quad (4.9)$$

Then

$$\tilde{D}_a(|\alpha\rangle, |\beta\rangle) = \frac{1}{\sqrt{2}}|\alpha - \beta|\sqrt{1 + \exp\,(-|\alpha - \beta|^2)}. \quad (4.10)$$

The right-hand side of eq. (4.10) is a monotonous function of $|\alpha - \beta|$, increasing from $|\alpha - \beta|$ at $|\alpha - \beta| \ll 1$ to $|\alpha - \beta|/\sqrt{2}$ at $|\alpha - \beta| \gg 1$. Although we have no proof that the quasidistance $\tilde{D}_a$ satisfies the triangle inequality (2.3) for *all* states, we can prove that the function (4.10) satisfies this inequality for all values of α and β.

5. Examples

5.1. *Coherent and Fock's states*

For two *coherent states* $|\alpha\rangle$ and $|\beta\rangle$ one finds

$$d(|\alpha\rangle, |\beta\rangle) = \sqrt{2}\,[1 - \exp\,(-|\alpha - \beta|^2)]^{1/2}. \quad (5.1)$$

If $|\alpha - \beta| \ll 1$, then $d(|\alpha\rangle, |\beta\rangle) \approx \sqrt{2}|\alpha - \beta|$ is proportional to the geometric distance of the displacement parameters α and β in the complex plane, but it goes to $\sqrt{2}$ when $|\alpha - \beta| \gg 1$. The N-distance (4.3) between the coherent states is given by

$$d_N(|\alpha\rangle, |\beta\rangle)$$
$$= [|\alpha|^2 + |\beta|^2 - 2\,\mathrm{Re}\,(\beta^*\alpha)\exp\,(-|\alpha - \beta|^2)]^{1/2} \quad (5.2)$$

so $d_N(|\alpha\rangle, |0\rangle) > d_N(|\beta\rangle, |0\rangle)$ if $|\alpha| > |\beta|$. The N-distance is equal to the geometrical distance $|\alpha - \beta|$ in the complex plane of parameters, if $\mathrm{Re}\,(\alpha\beta^*) = 0$ (i.e. for orthogonal directions in the complex plane). In Fig. 1 we plot the HS- and N-distances between the Fock state $|m\rangle$ and the coherent state $|\alpha\rangle$

$$d^{(\mathrm{HS})}(|\alpha\rangle, |m\rangle) = \sqrt{2}\left(1 - \frac{|\alpha|^{2m}}{m!}\,\mathrm{e}^{-|\alpha|^2}\right)^{1/2}, \quad (5.3)$$

$$d_N(|\alpha\rangle, |m\rangle) = \left(m + |\alpha|^2 - \frac{2|\alpha|^{2m}}{(m-1)!}\,\mathrm{e}^{-|\alpha|^2}\right)^{1/2} \quad (5.4)$$

as functions of the mean photon number $|\alpha|^2$ for fixed values of $m = 1, 2, 3$. The HS-distance has a minimum at $|\alpha|^2 = m$. For small values of $|\alpha|^2$ we have $d^{(\mathrm{HS})}(|\alpha\rangle, |m\rangle) > d^{(\mathrm{HS})}(|\alpha\rangle, |n\rangle)$ if $m > n$, but this inequality changes its sign if $|\alpha|^2$ is sufficiently large. The N-distance

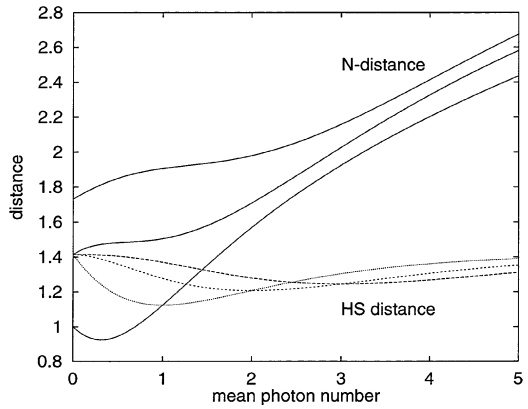

Fig. 1. The dependences of the N-distance (three upper curves) and the Hilbert–Schmidt distance (three lower curves) between the coherent state $|\alpha\rangle$ and the Fock states $|m\rangle$ with $m = 1, 2, 3$, versus the mean photon number in the coherent state $|\alpha|^2$. The order of curves from bottom to top (in the part of plot nearby the vertical axis): the lower curves correspond to $m = 1$ while the upper ones correspond to $m = 3$.

has a minimum only for $m = 1$, and the m-dependence is monotonous for all values of $|\alpha|^2$.

5.2. *Squeezed vacuum states*

The *squeezed vacuum state* [53] depends on the complex parameter ζ with $|\zeta| < 1$

$$|\zeta\rangle = (1 - |\zeta|^2)^{1/4} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} \zeta^n |2n\rangle. \tag{5.5}$$

The HS-distance between the states $|\zeta_1\rangle$ and $|\zeta_2\rangle$ reads (see also [50, 54])

$$d(|\zeta_1\rangle, |\zeta_2\rangle) = \frac{\sqrt{2}\,|\zeta_1 - \zeta_2|}{|1 - \zeta_1\zeta_2^*|^{1/2}} [|1 - \zeta_1\zeta_2^*| + \sqrt{(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)}]^{-1/2}. \tag{5.6}$$

For $|\zeta_1| \ll 1$ and $|\zeta_2| \ll 1$ this is the geometric distance of the complex squeezing parameters. Using the parametrisation $\zeta = \tanh \tau\, e^{i\phi}$, $\tau \geqslant 0$, we have a simplified formula in the case $\phi_1 = \phi_2$:

$$d(|\zeta_1\rangle, |\zeta_2\rangle) = \frac{2 \sinh [\frac{1}{2}(\tau_1 - \tau_2)]}{\sqrt{\cosh (\tau_1 - \tau_2)}}. \tag{5.7}$$

For $\tau_2 = 0$ (5.7) gives the distance between the vacuum and the squeezed state $|\zeta_1\rangle$.

The N-distance can be expressed as

$$d_N^2(|\zeta_1\rangle, |\zeta_2\rangle) = \frac{|\zeta_1|^2}{1 - |\zeta_1|^2} + \frac{|\zeta_2|^2}{1 - |\zeta_2|^2} + 2\,\frac{|\zeta_1\zeta_2|^2 - \mathrm{Re}\,(\zeta_1\zeta_2^*)}{|1 - \zeta_1\zeta_2^*|^3} \times \sqrt{(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)}. \tag{5.8}$$

If $|\zeta_{1,2}| \ll 1$, then (5.8) has the same limit as the "unpolarized" Hilbert–Schmidt distance (5.6): $d_N \approx d \approx |\zeta_1 - \zeta_2|$. However, for large values of the squeezing parameter these two distances become completely different. For example, in the special case $\Delta\phi \equiv \arg \zeta_1 - \arg \zeta_2 = 0$ we have instead of (5.7) the expression ($\tau_j \equiv |\zeta_j|$)

$$d_N^2(|\zeta_1\rangle, |\zeta_2\rangle) = \sinh^2\tau_1 + \sinh^2 \tau_2 - \frac{2 \sinh \tau_1 \sinh \tau_2}{\cosh^2 (\tau_1 - \tau_2)} \tag{5.9}$$

and $d_N (|\zeta\rangle, |0\rangle) = \sinh \tau$.

5.3. *"Schrödinger cat" states*

Now let us consider the family of the "Schrödinger cat" states

$$|\alpha; \varphi\rangle = (2[1 + \cos \varphi \exp (-2|\alpha|^2)])^{-1/2}(|\alpha\rangle + e^{i\varphi}|-\alpha\rangle). \tag{5.10}$$

The special cases of this family are even states ($\varphi = 0$), odd states ($\varphi = \pi$) and the Yurke–Stoler states ($\varphi = \pi/2$). A more general set of states $|\alpha;\ \tau,\ \varphi\rangle \sim |\alpha\rangle + \tau e^{i\varphi}|-\alpha\rangle$ was studied in [55]. The square of the distance between the coherent and cat state with the same values of the parameter α equals

$$d^2(|\alpha; \varphi\rangle, |\alpha\rangle) = \frac{1 - \exp (-4|\alpha|^2)}{1 + \cos \varphi \exp (-2|\alpha|^2)}. \tag{5.11}$$

For the distance from the vacuum state we obtain

$$d^2(|\alpha; \varphi\rangle, |0\rangle) = \frac{2[1 - \exp (-|\alpha|^2)]}{1 + \cos \varphi \exp (-2|\alpha|^2)} \tag{5.12}$$

whereas the distance between the two states with the same parameter α but different values of phases φ_1 and φ_2 reads

$$d^2(\varphi_1, \varphi_2) = [1 - \exp (-4|\alpha|^2)][1 - \cos (\varphi_1 - \varphi_2)] \times [1 + \cos \varphi_1 \exp (-2|\alpha|^2)]^{-1} \times [1 + \cos \varphi_2 \exp (-2|\alpha|^2)]^{-1}. \tag{5.13}$$

For $|\alpha| \gg 1$ we have $d^2(\varphi_1, \varphi_2) \approx 2 \sin^2 (|\varphi_1 - \varphi_2|/2)$.

The N-distances between the same states have an extra factor $|\alpha|$:

$$d_N^2(|\alpha; \varphi\rangle, |0\rangle) = |\alpha|^2 \frac{1 - \cos \varphi \exp (-2|\alpha|^2)}{1 + \cos \varphi \exp (-2|\alpha|^2)}, \tag{5.14}$$

$$d_N^2(\varphi_1, \varphi_2) = |\alpha|^2[1 + \exp (-4|\alpha|^2)][1 - \cos (\varphi_1 - \varphi_2)] \times [1 + \cos \varphi_1 \exp (-2|\alpha|^2)]^{-1} \times [1 + \cos \varphi_2 \exp (-2|\alpha|^2)]^{-1}. \tag{5.15}$$

Now we have $d_N \approx \sqrt{2}|\alpha| \sin (|\varphi_1 - \varphi_2|/2)$ for $|\alpha| \gg 1$.

Equations (5.11)–(5.15) clearly show that the YS-states are intermediate between even and odd ones. Moreover, we see that the distance between the YS and the odd states with the same $|\alpha|$ is greater than that between the YS and the even states, and the YS-state is farther from the coherent state than the even state (whereas the Mandel parameter does not distinguish the coherent and YS states at all). This example demonstrates how the concept of distance helps to understand better the properties of different families of quantum states and the mutual relations between them.

5.4. *Coherent phase states*

As a further example we consider the *coherent phase states* [56]

$$|\varepsilon\rangle = \sqrt{1 - \varepsilon\varepsilon^*} \sum_{n=0}^{\infty} \varepsilon^n |n\rangle, \quad \hat{E}_-|\varepsilon\rangle = \varepsilon|\varepsilon\rangle, \quad |\varepsilon| < 1 \tag{5.16}$$

where

$$\hat{E}_- \equiv \sum_{n=1}^{\infty} |n-1\rangle\langle n| = (\hat{a}\hat{a}\dagger)^{-1/2}\hat{a}$$

is the Susskind–Glogower phase operator [57] which can be considered to certain extent as a quantum analogue of the

classical phase $e^{i\varphi}$. The HS distance between the states $|\varepsilon_1\rangle$ and $|\varepsilon_2\rangle$ is given by

$$d(|\varepsilon_1\rangle, |\varepsilon_2\rangle) = \frac{\sqrt{2}|\varepsilon_1 - \varepsilon_2|}{|1 - \varepsilon_1\varepsilon_2^*|}. \tag{5.17}$$

It is proportional to the geometric distance of the complex parameters ε_1 and ε_2 for $|\varepsilon_{1,2}| \ll 1$. For any $|\varepsilon| < 1$ the distance from the vacuum state is simply $d(|\varepsilon\rangle, |0\rangle) = \sqrt{2}|\varepsilon|$. At the same time, the d_N-distance is given by

$$d_N^2(|\varepsilon_1\rangle, |\varepsilon_2\rangle) = \frac{|\varepsilon_1|^2}{1 - |\varepsilon_1|^2} + \frac{|\varepsilon_2|^2}{1 - |\varepsilon_2|^2} + 2\frac{(1 - |\varepsilon_1|^2)(1 - |\varepsilon_2|^2)[|\varepsilon_1\varepsilon_2|^2 - \mathrm{Re}\,(\varepsilon_1\varepsilon_2^*)]}{[1 - 2\,\mathrm{Re}\,(\varepsilon_1\varepsilon_2^*) + |\varepsilon_1\varepsilon_2|^2]^2}. \tag{5.18}$$

In particular, $d_N(|\varepsilon\rangle, |0\rangle) = |\varepsilon|(1 - |\varepsilon|^2)^{-1/2}$.

5.5. *Thermal states*

The pure quantum state (5.16) has the same probability distribution $|\langle n|\varepsilon\rangle|^2$ as the mixed *thermal state* described by the statistical operator

$$\hat{\rho} = \frac{1}{1 + \bar{n}} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{1 + \bar{n}}\right)^n |n\rangle\langle n| \tag{5.19}$$

provided that one identifies the mean photon number $\bar{n}$ with $|\varepsilon|^2/(1 - |\varepsilon|^2)$ [58]. Moreover, the state (5.16) arises naturally as an exact solution to some nonlinear modifications of the Schrödinger equation [59], so it can be named also a "pseudothermal state" [59]. Therefore it is interesting to compare the expressions (5.17) and (5.18) for the distances between "pseudothermal" states with the analogous formulae for the true thermal states.

The HS distance between two states (5.19) reads

$$d^{(\mathrm{HS})}(\bar{n}_1, \bar{n}_2) = \frac{\sqrt{2}|\bar{n}_1 - \bar{n}_2|}{\sqrt{(1 + 2\bar{n}_1)(1 + 2\bar{n}_2)(1 + \bar{n}_1 + \bar{n}_2)}}. \tag{5.20}$$

Although it is proportional to the difference of the mean photon numbers, it goes to zero when $\bar{n}_{1,2} \to \infty$ and $|\bar{n}_1 - \bar{n}_2| = const$. The distance to the ground state equals

$$d^{(\mathrm{HS})}(\bar{n}, 0) = \frac{\bar{n}\sqrt{2}}{\sqrt{(1 + \bar{n})(1 + 2\bar{n})}}, \tag{5.21}$$

and it tends to 1 when $\bar{n} \to \infty$, i.e. to the value which is $\sqrt{2}$ times less than the maximal possible Hilbert–Schmidt distance. These results become clear if one remembers that highly mixed states are located, in a sense, deeply "inside" the Hilbert space, since the density operators form a convex set with the pure states contained in the boundary [60]. Nonetheless, being justified from the mathematical point of view, these properties do not agree completely with our physical intuition, because usually we think on highly mixed states as almost classical ones (all the coherence is lost), which must be far away from the intrinsically quantum vacuum state. In particular, it seems a little bit strange that high temperature states are closer to the ground one than any pure Fock state.

Using the modified HS distance (3.13) (which coincides with the Bures–Uhlmann distance in the case involved) we obtain

$$d^{(\mathrm{BU})}(\bar{n}_1, \bar{n}_2) = \sqrt{2}\left[1 - \frac{\sqrt{(1 + \bar{n}_1)(1 + \bar{n}_2)} + \sqrt{\bar{n}_1\bar{n}_2}}{1 + \bar{n}_1 + \bar{n}_2}\right]^{1/2}. \tag{5.22}$$

In particular, the distance to the ground state equals

$$d^{(\mathrm{BU})}(\bar{n}, 0) = \frac{\sqrt{2\bar{n}}}{[\sqrt{1 + \bar{n}}(1 + \sqrt{1 + \bar{n}})]^{1/2}} \tag{5.23}$$

and it tends to the maximal possible value $\sqrt{2}$ when $\bar{n} \to \infty$. It is interesting to compare this formula with the analogous one for the "pseudothermal" state (5.16), but written in terms of the mean photon number:

$$d^{(\mathrm{HS})}(|\varepsilon\rangle, 0) = \sqrt{\frac{2\bar{n}}{1 + \bar{n}}}.$$

We see that the BU-distance for the mixed states is always a little bit less than the distance between the vacuum and the pure pseudothermal state with the same value of $\bar{n}$, in agreement with the reasonings of the preceding paragraph. For $\bar{n}_{1,2} \gg 1$ (5.22) is simplified

$$d^{(\mathrm{BU})}(\bar{n}_1, \bar{n}_2) \approx \frac{\sqrt{2}|\sqrt{\bar{n}_1} - \sqrt{\bar{n}_1}|}{\sqrt{\bar{n}_1 + \bar{n}_2}}. \tag{5.24}$$

The N-distance between two thermal states (5.19) reads

$$d_N(\bar{n}_1, \bar{n}_2) = \frac{|\bar{n}_1 - \bar{n}_2|\sqrt{(1 + \bar{n}_1 + \bar{n}_2)^2 + 2\bar{n}_1\bar{n}_2(1 + 2\bar{n}_1)(1 + 2\bar{n}_2)}}{(1 + 2\bar{n}_1)(1 + 2\bar{n}_2)(1 + \bar{n}_1 + \bar{n}_2)}. \tag{5.25}$$

As well as for the HS distance, the high temperature states occur not very far from the ground one:

$$d_N(\bar{n}, 0) = \frac{\bar{n}}{1 - 2\bar{n}} \to \tfrac{1}{2} \quad \text{when } \bar{n} \to \infty.$$

At the same time, using the modified N-distance (4.2) we obtain the expression

$$\tilde{d}_N^2(\bar{n}_1, \bar{n}_2) = \bar{n}_1 + \bar{n}_2 - 2\sqrt{\bar{n}_1\bar{n}_2} \times \left(\frac{\sqrt{(1 + \bar{n}_1)(1 + \bar{n}_2)} + \sqrt{\bar{n}_1\bar{n}_2}}{1 + \bar{n}_1 + \bar{n}_2}\right)^2 \tag{5.26}$$

which yields $\tilde{d}_N(\bar{n}, 0) = \bar{n}^{1/2}$, as well as for pure states. Analysing formula (5.18) for the N-distance between the "pseudothermal" states, one can check that the right-hand side attains the minimum (for fixed absolute values $|\varepsilon_{1,2}|$) if $\mathrm{Re}\,(\varepsilon_1^*\varepsilon_2) = |\varepsilon_1\varepsilon_2|$. This minimal distance can be written in terms of $\bar{n}_{1,2}$ in the form very similar to (5.26), but the last factor has the exponent 3 instead of 2:

$$\tilde{d}_{N\,\min}^2(|\varepsilon_1\rangle, |\varepsilon_2\rangle) = \bar{n}_1 + \bar{n}_2 - 2\sqrt{\bar{n}_1\bar{n}_2} \times \left(\frac{\sqrt{(1 + \bar{n}_1)(1 + \bar{n}_2)} + \sqrt{\bar{n}_1\bar{n}_2}}{1 + \bar{n}_1 + \bar{n}_2}\right)^3. \tag{5.27}$$

Since the fraction inside the parentheses does not exceed 1 (this is a consequence of the inequality $\bar{n}_1 + \bar{n}_2 \geqslant 2\sqrt{\bar{n}_1\bar{n}_2}$), we have $\tilde{d}_{N\,\min}(|\varepsilon_1\rangle, |\varepsilon_2\rangle) \geqslant \tilde{d}_N(\bar{n}_1, \bar{n}_2)$ for any pair of pure and mixed states with the same mean photon numbers.

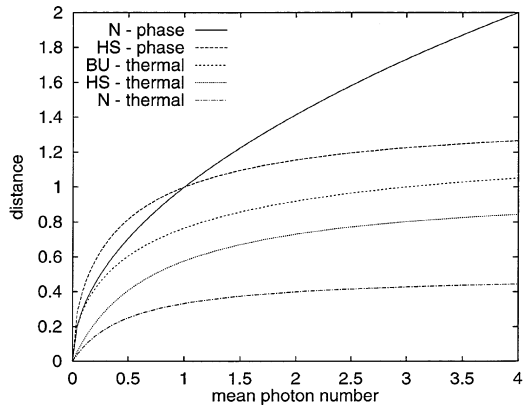

Fig. 2. Different distances between the vacuum and the thermal (mixed) and pseudothermal (pure phase coherent) states versus the mean photon number. The order of the curves in the right-hand side of the plot (from bottom to top): N-distance for the thermal state; the Hilbert–Schmidt distance for the thermal state; the Bures–Uhlmann distance for the thermal state; the Hilbert–Schmidt distance for the pseudothermal state; N-distance for the pseudothermal state (it coincides with the modified N-distance $\tilde{d}_N$ for the thermal state in the case concerned).

Equations (5.26) and (5.27) can be simplified for $\bar{n}_{1,2} \gg 1$:

$$\tilde{d}_N^2(\bar{n}_1, \bar{n}_2) \approx \bar{n}_1 + \bar{n}_2 - \frac{8(\bar{n}_1\bar{n}_2)^{3/2}}{(\bar{n}_1 + \bar{n}_2)^2},$$

$$\tilde{d}_{N\,\min}^2(|\varepsilon_1\rangle, |\varepsilon_2\rangle) \approx \bar{n}_1 + \bar{n}_2 - \frac{16(\bar{n}_1\bar{n}_2)^2}{(\bar{n}_1 + \bar{n}_2)^3}.$$

If also $|\bar{n}_1 - \bar{n}_2| \ll \bar{n}_{1,2}$, then we obtain approximate expressions resembling formula (4.8) for the *quasi*distance between the Fock states, but with different coefficients

$$\tilde{d}_N^2(\bar{n}_1, \bar{n}_2) \approx \sqrt{3}\,|\sqrt{\bar{n}_1} - \sqrt{\bar{n}_2}| = \frac{\sqrt{3}\,|\bar{n}_1 - \bar{n}_2|}{\sqrt{\bar{n}_1} + \sqrt{\bar{n}_2}},$$

$$\tilde{d}_{N\,\min}(|\varepsilon_1\rangle, |\varepsilon_2\rangle) \approx 2\,|\sqrt{\bar{n}_1} - \sqrt{\bar{n}_2}| = \frac{2\,|\bar{n}_1 - \bar{n}_2|}{\sqrt{\bar{n}_1} + \sqrt{\bar{n}_2}}.$$

The dependences of different distances between the vacuum and thermal or "pseudothermal" states on the mean photon number $\bar{n}$ are shown in Fig. 2. The distances of the pure states are larger than analogous distances of the mixed states with the same mean photon numbers, excepting the case of the $\tilde{d}_N$-distance which is the same both for the thermal and the phase coherent states. We may conclude that the $\tilde{d}_N$-distance seems to be the most adequate from the physical point of view (at least for thermal states).

6. "Classical-like" quantum distances

It is accepted that quantum states are described in terms of the wave functions (state vectors in the Hilbert space) or density matrices (statistical operators). However, these *complex-valued* objects have rather indirect relations to the results of measurements, which are expressed in terms of real positive probabilities. Recently, a new formulation of quantum mechanics in terms of positive classical probability distributions was proposed [6, 61, 62]. It is a natural consequence of the concepts of the so called *symplectic tomography* developed in [63, 64].

Let us introduce the two-parameter family of quadrature operators $\hat{X}_{\mu\nu} = \mu\hat{q} + \nu\hat{p}$, $-\infty < \mu, \nu < \infty$, where $\hat{q}$ and $\hat{p}$ are the usual coordinate and momentum operators (in one dimension for simplicity). It can be shown that the probability distribution $w_{\mu\nu}(X)$ of the real eigenvalues of the Hermitian operator $\hat{X}_{\mu\nu}$ is given by the following integral transform of the Wigner function:

$$w_{\mu\nu}(X) = \int \frac{\mathrm{d}q\,\mathrm{d}p}{2\pi}\,\delta(\mu q + \nu p - X)W(q, p). \tag{6.1}$$

The reciprocal transform

$$W(q, p) = \frac{1}{2\pi}\int \mathrm{d}X\,\mathrm{d}\mu\,\mathrm{d}\nu\,\exp\,[\mathrm{i}(X - \mu q - \nu p)]w_{\mu\nu}(X) \tag{6.2}$$

enables to express any Wigner function (and, consequently, any density matrix) in terms of the positive marginal probability distributions $w_{\mu\nu}(X)$ which can be obtained, in principle, directly from an experiment with the aid of the homodyne detection schemes. Consequently, the description in terms of the family of classical distributions $w_{\mu\nu}(X)$ is completely equivalent to the standard description in terms of the density matrix or the wave function. This fact is the basis of the "classical-like" formulation of quantum mechanics [6, 61, 62, 65, 66, 67]. In this formulation every quantum state is described not by a single complex-valued function $\psi(x)$ or $\rho(x, x')$, but by an infinite set of *classical positive probability distributions* $w_{\mu\nu}(X)$, $-\infty < \mu, \nu < \infty$. For example, the Fock state of the harmonic oscillator $|n\rangle$ is described by the family of the marginal distributions [61]

$$w_{\mu\nu}^{(n)}(X) = w_{\mu\nu}^{(0)}(X)\,\frac{1}{2^n n!}\,H_n^2\left(\frac{X}{\sqrt{\mu^2 + \nu^2}}\right) \tag{6.3}$$

where $H_n(z)$ is the Hermite polynomial, while the marginal distribution $w_{\mu\nu}^{(0)}(X)$ of the vacuum state reads

$$w_{\mu\nu}^{(0)}(X) = \frac{1}{\sqrt{\pi(\mu^2 + \nu^2)}}\,\exp\left(-\frac{X^2}{\mu^2 + \nu^2}\right). \tag{6.4}$$

Now, considering the quantum states described by two different sets of the marginal distributions $w_{\mu\nu}^{(a)}(X)$ and $w_{\mu\nu}^{(b)}(X)$ we may define the "classical-like" distance between these states as

$$\mathscr{D}_{ab}^{\mathscr{C}} = \int \mathrm{d}\mu\,\mathrm{d}\nu g(\mu, \nu) d_{ab}^{(\mathscr{C})}(w_{\mu\nu}^{(a)}, w_{\mu\nu}^{(b)}) \tag{6.5}$$

where $d_{ab}^{\mathscr{C}}(w_{\mu\nu}^{(a)}, w_{\mu\nu}^{(b)})$ is some *classical* distance between the distributions $w_{\mu\nu}^{(a)}(X)$ and $w_{\mu\nu}^{(b)}(X)$. A positive weight function $g(\mu, \nu)$ is introduced to ensure the convergence of the integral over μ, ν. Evidently, if the "partial distance" $d_{ab}^{\mathscr{C}}(w_{\mu\nu}^{(a)}, w_{\mu\nu}^{(b)})$ satisfies the triangle inequality for all fixed values of μ, ν, this inequality remains valid after multiplying by the positive function $g(\mu, \nu)$ and the subsequent integration over μ, ν.

Let us consider, for example, the "Kakutani–Hellinger–Matusita distance" [68, 69] between two real nonnegative distributions $P_1(x)$ and $P_2(x)$

$$d_{\mathscr{H}}(P_1, P_2) = \left[\int \mathrm{d}x(\sqrt{P_1(x)} - \sqrt{P_2(x)})^2\right]^{1/2}. \tag{6.6}$$

Taking into account the normalization condition we arrive at the "classical-like" analogue of the Bures–Uhlmann distance

$$\mathscr{D}_{ab}^{\mathscr{H}} = \sqrt{2}\int \mathrm{d}\mu\,\mathrm{d}\nu g(\mu, \nu) \times \left[1 - \int \mathrm{d}X\sqrt{w_{\mu\nu}^{(a)}(X)w_{\mu\nu}^{(b)}(X)}\right]^{1/2}. \tag{6.7}$$

The "classical-like" analogue of the JMG-distance (2.8) is obtained if one chooses for $d_{ab}^{\mathscr{C}}$ the classical Kolmogorov distance [68]

$$d_{\mathscr{K}}(P_1, P_2) = \int \mathrm{d}x\,|P_1(x) - P_2(x)|. \tag{6.8}$$

To illustrate the new approach let us consider the $\mathscr{D}^{\mathscr{H}}$-distance (6.7) between two coherent states $|\alpha\rangle$ and $|\beta\rangle$. Each of these states is described by means of the families of the marginal distributions like

$$w_{\mu\nu}^{(\alpha)}(X) = \frac{1}{\sqrt{\pi(\mu^2+\nu^2)}} \exp\left(-\frac{[X - \bar{X}_\alpha(\mu, \nu)]^2}{\mu^2+\nu^2}\right),$$
$$\bar{X}_\alpha(\mu, \nu) = \sqrt{2}(\mu \operatorname{Re} \alpha + \nu \operatorname{Im} \alpha). \tag{6.9}$$

Introducing the polar coordinates in the $\mu\nu$ plane, $\mu = R\cos\vartheta$, $\nu = R\sin\vartheta$, we see that the $\mathscr{D}^{\mathscr{H}}$-distance between the coherent states depends on $|\alpha - \beta|$ only:

$$\mathscr{D}_{\alpha\beta}^{\mathscr{H}} = \int_0^\infty R\,\mathrm{d}R \int_0^{2\pi} \mathrm{d}\vartheta g(R, \vartheta)$$
$$\times \{2 - 2\exp[-\tfrac{1}{2}|\alpha-\beta|^2\cos^2(\vartheta - \varphi)]\}^{1/2} \tag{6.10}$$

(here φ is the phase of the complex number $\alpha - \beta$). It is convenient to choose the weight function $g(R, \vartheta)$ independent on ϑ and to impose the condition $\int_0^\infty g(R)R\,\mathrm{d}R = 1$. Then for close coherent states we have $\mathscr{D}_{\alpha\beta}^{\mathscr{H}} = 4|\alpha - \beta|$ if $|\alpha - \beta| \ll 1$. When $|\alpha - \beta| \to \infty$, the $\mathscr{D}^{\mathscr{H}}$-distance tends to the constant value $2\pi\sqrt{2}$.

The integral over μ, ν can be calculated explicitly for classical-like *distinguishability measures* (DM) which are defined by the same formula (6.5) but without imposing the requirement (2.3) (the triangle inequality) on the function $d_{ab}^{\mathscr{C}}(w_{\mu\nu}^{(a)}, w_{\mu\nu}^{(b)})$. The distinguishability measures are frequently used in the classical statistics and the information theory [68]. Their applications to quantum mechanical problems were discussed recently in [70, 71]. The most known examples of classical DM are the *Bhattacharyya coefficient* [68]

$$\mathscr{B}(P_1, P_2) = -\ln \int \mathrm{d}x\sqrt{P_1(x)P_2(x)} \tag{6.11}$$

and the *Kullback–Liebler distinguishability measure* [68]

$$\mathscr{I}(P_1, P_2) = \int \mathrm{d}x[P_1(x) - P_2(x)] \ln \frac{P_1(x)}{P_2(x)}. \tag{6.12}$$

For coherent states both these measures yield similar dependences on the parameters α and β, which differ only in a scale factor (we assume the same weight function $g(\mu, \nu)$ as above):

$$\mathscr{D}_{\alpha\beta}^{(\mathscr{I})} = 8\mathscr{D}_{\alpha\beta}^{(\mathscr{B})} = 4\pi|\alpha - \beta|^2. \tag{6.13}$$

These quantum DM are unbounded when $|\alpha - \beta| \to \infty$, but they do not satisfy the triangle inequality.

7. Conclusion

Let us summarise the main results of the paper. We have obtained new inequalities for the Hilbert–Schmidt distance and its modifications, which can be used for evaluating the "degree of proximity" between close quantum states. We have given new expressions for the Hilbert–Schmidt distance in terms of quasiprobability distributions and in terms of the ordered moments. We have constructed the distances which are sensitive to the energy of quantum states. These "N-distances" are unlimited and they distinguish different orthogonal states. Besides, we have shown how the concept of distance can be introduced in the framework of the new "classical-like" formulation of quantum mechanics in terms of positive probability distributions of the rotated (in the phase space) quadrature operators.

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