# Minimal sets of dequantizers and quantizers for finite-dimensional quantum systems 

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#### Abstract

The problem of finding and characterizing minimal sets of dequantizers and quantizers applied in the mapping of operators onto functions is considered, for finite-dimensional quantum systems. The general properties of such sets are determined. An explicit description of all the minimum self-dual sets of dequantizers and quantizers for a qubit system is derived.


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## 1. Introduction

The phase-space formulation of quantum mechanics is still in the focus of research interest, as it has numerous important applications [1-4]. Quasi-probability distributions such as the Wigner function [5], the Husimi Q-function [6,7] and Glauber-Sudarshan P-function [8,9] describe completely the states of a quantum system and they are widely used for calculations in various physical problems [10-15]. They have proven to be very useful in quantum optics [16-18]. A probability representation with fair probability distributions defined on the phase space has also been introduced in the literature [19-21]. A probability distribution called the symplectic tomogram was introduced in connection with measuring the quantum states of light by means of optical homodyne tomography [22-24]. The properties of this tomographic probability representation are discussed in detail in review [25].

In order to use quasi-probability distributions and tomograms in physical problems the operators modeling observable physical quantities have to be represented. [26]. This representation is called the symbol of operators. The algebra of symbols corresponding all possible manipulations with operators on the Hilbert space can be constructed by applying the general star-product scheme

[^0][27-29]. Within this formalism one can relate operators to their symbols using dequantizers and can reconstruct operators from their symbols using quantizers. The relations between different phase-space representations can be also determined in this framework [29-32].

All these ideas can be extended to finite dimensional quantum systems. Finding a complete, continuous Wigner function for such system is still a subject of investigations [33-35]. Beside these efforts there is an increasing interest in the construction of discrete phase spaces and Wigner functions owing to their possible applications in quantum information science. There are several ways of constructing such a phase space and the definition of a discrete Wigner function in this space is still ambiguous [36-47]. The approach introduced in [40] has proven to be well suited to study various quantum information problems [48]. In this method, an $N \times N$ phase space is defined for $N$ dimensional quantum systems, where $N$ is a power of a prime number. This is the case, e.g. for qubit systems. This phase space has the same geometric properties as those of the ordinary infinite dimensional phase space. Wigner functions can be defined in this space using Hermitian operators connected to special mutually orthogonal sets of parallel lines called striations. There exist $N+1$ different striations and the bases associated with them are mutually unbiased $[4,43,49,50]$. Such discrete Wigner functions have the same essential properties as their continuous counterparts. The most interesting one from the point of view of tomographic measurements is that the sum of
values of a Wigner function along any line in phase space is equal to the probability of detecting the basis state associated with the line [48].

Tomographic probability distributions called spin tomograms [51-54], and unitary matrix tomograms [55] have been also developed for finite dimensional spin systems. The star product formalism of symbols for $N$-dimensional systems is described in detail in [56]. Using this formalism the relations between tomograms and Wigner functions for one and two qubits have been determined [57,58,56].

In this paper we consider the problem of finding and characterizing minimal sets of quantizers and dequantizers for finite dimensional quantum systems. We determine the general properties of such sets. Given minimal sets of dequantizers and quantizers for a particular quantum system, any type of symbols of the operators and the quantum states consisting of minimal elements, e.g., discrete Wigner functions, can be treated in a common framework. We find explicit expressions describing all the minimal self-dual sets of dequantizers and quantizers for a qubit system.

The paper is organized as follows. In Section 2 we present the general formalism of mapping operators onto functions based on dequantizers and quantizers. The general properties of minimal sets of dequantizers and quantizers for $N$ dimensional systems is described in Section 3. In Section 4 the explicit form of all minimal self-dual sets of dequantizers and quantizers for a qubit system is found.

## 2. Dequantizers and quantizers

In this section we summarize the general formalism of using $c$-number functions instead of operators to describe quantum systems [26-29]. Let $\hat{A}$ be a Hermitian operator acting on a Hilbert space $\mathcal{H}$ so it can be an operator describing an observable or the density operator $\hat{\rho}$ of the quantum system. Suppose we have a set of linear operators $\hat{U}(x)$ acting on $\mathcal{H}$ and labelled by the parameter $x$ that is an $n$-dimensional vector $x=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ in the general case. One can construct a $c$-number function $f_{\hat{A}}(x)$ called the symbol of the operator $\hat{A}$ using the definition
$f_{\hat{A}}(x)=\operatorname{Tr}[\hat{A} \hat{U}(x)]$.
This linear mapping of operators onto functions is invertible if there is a set of operators $\hat{D}(x)$ acting on $\mathcal{H}$ such that
$\hat{A}=\int f_{\hat{A}}(x) \hat{D}(x) \mathrm{d} x$.
The operators $\hat{U}(x)$ and $\hat{D}(x)$ are called dequantizers and quantizers, respectively. In this formalism the operation for functions corresponding to the multiplication of $\hat{A}$ and $\hat{B}$ is called star product and defined by
$f_{\hat{A} \hat{B}}(x)=f_{\hat{A}}(x) * f_{\hat{B}}(x)=\operatorname{Tr}[\hat{A} \hat{B} \hat{U}(x)]$.
Multiplying Eq. (2) by the operator $\hat{U}\left(x^{\prime}\right)$ and taking the trace we get
$f_{\hat{A}}\left(x^{\prime}\right)=\int f_{\hat{A}}(x) \operatorname{Tr}\left[\hat{D}(x) \hat{U}\left(x^{\prime}\right)\right] \mathrm{d} x$.
For continuous systems the operators $\hat{U}(x)$ are defined in the usual phase space with the coordinates ( $q, p$ ) while for discrete systems $x$ can be both discrete and continuous as in the case of spin tomograms, or it can be purely discrete as in the case of discrete Wigner functions defined e.g. in [40]. In the latter case Eqs. (2) and (4) can be written as
$\hat{A}=\sum_{k=1}^{N} f_{\hat{A}}(k) \hat{D}(k)$
and
$f_{\hat{A}}\left(k^{\prime}\right)=\sum_{k=1}^{N} f_{\hat{A}}(k) \operatorname{Tr}\left[\hat{D}(k) \hat{U}\left(k^{\prime}\right)\right]$,
respectively.
For a $d$ dimensional discrete quantum system the term minimal set of quantizers and dequantizers is introduced for sets containing $d^{2}$ linearly independent operators. From Eq. (6) it follows that the quantizer and dequantizer operators of such sets satisfy the condition
$\operatorname{Tr}\left(\hat{D}(k) \hat{U}\left(k^{\prime}\right)\right)=\delta\left(k, k^{\prime}\right)$.
For some special set of dequantizers the symbols are called the Wigner function [40]. These dequantizers are Hermitian operators and coincide with the corresponding quantizers. So they form a self-dual system.

## 3. Minimal sets of dequantizers and quantizers

In this section we consider the general properties of minimal sets of quantizers and dequantizers for N -dimensional systems.

Let us analyze first a two-dimensional qubit system. For this system the minimal set of dequantizers consists of four linearly independent operators $\hat{U}^{(k)}$ that can be represented by four matrices
$\hat{U}^{(k)}=\left(\begin{array}{cc}U_{11}^{(k)} & U_{12}^{(k)} \\ U_{21}^{(k)} & U_{22}^{(k)}\end{array}\right), \quad k=1,2,3,4$.
First we address the problem of determining the four corresponding quantizers
$\hat{D}^{(k)}=\left(\begin{array}{ll}D_{11}^{(k)} & D_{12}^{(k)} \\ D_{21}^{(k)} & D_{22}^{(k)}\end{array}\right), \quad k=1,2,3,4$
satisfying Eq. (7) that can be written using the notations of Eqs. (8) and (9) as
$\operatorname{Tr}\left(\hat{U}^{(k)} \hat{D}^{\left(k^{\prime}\right)}\right)=\delta\left(k, k^{\prime}\right)$.
We assume that the dequantizers $U^{(k)}$ are known.
Let us introduce the operator
$\hat{A}=\left(\begin{array}{llll}U_{11}^{(1)} & U_{21}^{(1)} & U_{12}^{(1)} & U_{22}^{(1)} \\ U_{11}^{(2)} & U_{21}^{(2)} & U_{12}^{(2)} & U_{22}^{(2)} \\ U_{11}^{(3)} & U_{21}^{(3)} & U_{12}^{(3)} & U_{22}^{(3)} \\ U_{11}^{(4)} & U_{21}^{(4)} & U_{12}^{(4)} & U_{22}^{(4)}\end{array}\right), \quad m, n=1,2,3,4$
built up from the elements of the four dequantizer operators and the operator
$\hat{B}=\left(\begin{array}{llll}D_{11}^{(1)} & D_{12}^{(1)} & D_{21}^{(1)} & D_{22}^{(1)} \\ D_{11}^{(2)} & D_{12}^{(2)} & D_{21}^{(2)} & D_{22}^{(2)} \\ D_{11}^{(3)} & D_{12}^{(3)} & D_{21}^{(3)} & D_{22}^{(3)} \\ D_{11}^{(4)} & D_{12}^{(4)} & D_{21}^{(4)} & D_{22}^{(4)}\end{array}\right)$
containing the elements of the four quantizer operators. It is easy to see that the equation (10) is equivalent to
$\hat{A} \hat{B}^{T}=\hat{I}$
As the operators $\hat{U}^{(k)}$ are linearly independent therefore the determinant of the matrix $\hat{A}$ is not equal to zero. From Eq. (13) it
is clear that $\operatorname{det}(\hat{B})=(\operatorname{det}(\hat{A}))^{-1} \neq 0$ implying that the quantizers $\hat{D}^{(k)}$ are also linearly independent. From Eq. (10) or by performing the matrix product in (13) one can achieve four systems of equations labeled by $k^{\prime}$ each of which containing four linear equations labeled by $k$ in the form
$U_{11}^{(k)} D_{11}^{\left(k^{\prime}\right)}+U_{21}^{(k)} D_{12}^{\left(k^{\prime}\right)}+U_{12}^{(k)} D_{21}^{\left(k^{\prime}\right)}+U_{22}^{(k)} D_{22}^{\left(k^{\prime}\right)}=\delta\left(k, k^{\prime}\right)$,

$$
\begin{equation*}
k, k^{\prime}=1,2,3,4 \tag{14}
\end{equation*}
$$

The solution of these systems of equations can be formulated as follows:
$\hat{B}=\hat{A}^{(c)}(\operatorname{det}(\hat{A}))^{-1}$,
where $\hat{A}^{(c)}$ is the cofactor matrix of $\hat{A}$.
For example the elements of the quantizer $D^{(1)}$ can be expressed as
$D_{11}^{(1)}=A_{11}^{(c)}(\operatorname{det}(\hat{A}))^{-1}, \quad D_{12}^{(1)}=A_{12}^{(c)}(\operatorname{det}(\hat{A}))^{-1}$,
$D_{21}^{(1)}=A_{13}^{(c)}(\operatorname{det}(\hat{A}))^{-1}, \quad D_{22}^{(1)}=A_{14}^{(c)}(\operatorname{det}(\hat{A}))^{-1}$.
In the following we examine the problem of finding different minimal sets of dequantizers $\hat{U}^{(k)}$ and quantizers $\hat{D}^{\left(k^{\prime}\right)}$. Given a set of $\hat{U}^{(k)}-s$, one can derive another set $\hat{V}^{(j)}$ of dequantizers by applying a non-degenerate linear transformation $\hat{L}$ :
$\hat{V}^{(j)}=\sum_{k=1}^{4} L_{j k} \hat{U}^{(k)}, \quad j=1,2,3,4$.
Denoting the corresponding set of quantizers by $\hat{E}^{\left(j^{\prime}\right)}$ the novel sets will satisfy the relation (10), that is,
$\operatorname{Tr}\left(\hat{V}^{(j)} \hat{E}^{\left(j^{\prime}\right)}\right)=\delta\left(j, j^{\prime}\right)$.
The operators $\hat{E}^{\left(j^{\prime}\right)}$ can be viewed as linear combinations of the previous quantizers $\hat{D}^{(k)}$, and thus written as
$\hat{E}^{\left(j^{\prime}\right)}=\sum_{k=1}^{4} M_{j^{\prime} k} \hat{D}^{(k)}, \quad j^{\prime}=1,2,3,4$.
Substituting (17) and (19) into (18) the following matrix equation can be obtained for the four-dimensional $\hat{L}=\left\|L_{j k}\right\|$ and $\hat{M}=$ $\left\|M_{j^{\prime} k}\right\|$ matrices
$\hat{L} \hat{M}^{T}=\hat{I}$
Similarly to the case of Eq. (13), this equation leads to four systems of equations each of which contains four linear equations in the form

$$
\begin{equation*}
L_{j 1} M_{j^{\prime} 1}+L_{j 2} M_{j^{\prime} 2}+L_{j 3} M_{j^{\prime} 3}+L_{j 4} M_{j^{\prime} 4}=\delta\left(j, j^{\prime}\right) \tag{21}
\end{equation*}
$$

Solving these equations the matrix $\hat{M}$ can be written in the form
$\hat{M}=\hat{L}^{(c)}(\operatorname{det}(\hat{L}))^{-1}$
where $\hat{L}^{(c)}$ is the cofactor matrix of $\hat{L}$. It follows from Eq. (20) that if $\hat{L}$ is non-degenerate then $\hat{M}$ is non-degenerate, $\operatorname{too}(\operatorname{det}(\hat{M})=$ $\left.(\operatorname{det}(\hat{L}))^{-1} \neq 0\right)$.

In conclusion, knowing a minimal set of dequantizers one can derive other sets by using Eq. (17), while Eqs. (15) and (22) can be applied to determine the corresponding minimal sets of quantizers.

From Eq. (14) an additional proposition can be deduced. In this expression the values of the matrix elements $U_{11}^{(k)}, D_{11}^{(k)}, U_{22}^{(k)}, D_{22}^{(k)}$, $P^{(k)}, Q^{(k)}$ are real. If the dequantizers $\hat{U}^{(k)}$ are Hermitian, that is, $U_{12}^{(k)}=U_{21}^{(k)}$, in order to ensure that the expression remains real, the equality $D_{12}^{\left(k^{\prime}\right)}=D_{21}^{\left(k^{\prime}\right)^{*}}$ must be satisfied. It means that the quantizers are Hermitian, too.

Let us consider the case when the dequantizers are orthogonal to each other, that is,
$\operatorname{Tr}\left(\hat{U}^{(k)} \hat{U}^{\left(k^{\prime}\right)}\right)=\delta\left(k, k^{\prime}\right)$.
Hence, the operators $\hat{U}^{(k)}$ form an orthogonal basis in the space of operators acting on the vectors of the Hilbert space of qubits. Comparing Eqs. (10) and (23) for a fixed set of dequantizers $\hat{U}^{(k)}$ it is obvious that the choice $\hat{D}^{\left(k^{\prime}\right)}=\hat{U}^{\left(k^{\prime}\right)}$ is the solution of Eq. (10), that is, the corresponding quantizers $\hat{D}^{(k)}$ coincide with dequantizers $\hat{U}^{(k)}$. Evidently, they are orthogonal to each other, that is, $\operatorname{Tr}\left(\hat{D}^{(k)} \hat{D}^{\left(k^{\prime}\right)}\right)=\delta\left(k, k^{\prime}\right)$. So the quantizers $\hat{U}^{(k)}$ and dequantizers $\hat{D}^{(k)}$ form a self-dual system.

Though all these results are formulated for a qubit, they can easily be generalized for a multi-qubit systems of higher dimensions. In $d$ dimensions the operators $\hat{A}$ and $\hat{B}$ corresponding to the operators Eqs. (11) and (12), respectively, can be defined by $d^{2} \times d^{2}$ matrices. The connection between $\hat{A}$ and $\hat{B}$ can be still described by Eq. (15). The transformation rules for $\hat{U}^{(k)}$ and $\hat{D}^{(k)}$ can be easily generalized and all the previously discussed properties of these operators remain valid for higher dimensions.

## 4. Self-dual systems

In this section we consider a qubit system and analyze in detail the case when the minimal sets of dequantizers and quantizers coincide with each other, that is, they form a self-dual system and these operators are Hermitian.

Let us take four general Hermitian operators
$\hat{U}^{(k)}=\hat{D}^{(k)}=\left(\begin{array}{cc}a_{k} & b_{k}-\mathrm{i} c_{k} \\ b_{k}+\mathrm{i} c_{k} & d_{k}\end{array}\right), \quad k=1,2,3,4$,
where the parameters $a_{k}, b_{k}, c_{k}$, and $d_{k}$ are real. Our aim is to find explicitly the matrix elements of the operators $\hat{U}^{(k)}$ so that these matrices obey Eq. (23).

From Eq. (23) one can derive ten algebraic equations for the 16 parameters of the operators in (24). For different matrices, that is, $k \neq k^{\prime}$ we get

$$
\begin{align*}
& a_{1} a_{2}+d_{1} d_{2}+2\left(b_{1} b_{2}+c_{1} c_{2}\right)=0 \\
& a_{1} a_{3}+d_{1} d_{3}+2\left(b_{1} b_{3}+c_{1} c_{3}\right)=0 \\
& a_{1} a_{4}+d_{1} d_{4}+2\left(b_{1} b_{4}+c_{1} c_{4}\right)=0 \\
& a_{2} a_{3}+d_{2} d_{3}+2\left(b_{2} b_{3}+c_{2} c_{3}\right)=0  \tag{25}\\
& a_{2} a_{4}+d_{2} d_{4}+2\left(b_{2} b_{4}+c_{2} c_{4}\right)=0 \\
& a_{3} a_{4}+d_{3} d_{4}+2\left(b_{3} b_{4}+c_{3} c_{4}\right)=0
\end{align*}
$$

and for $k=k^{\prime}$ we have the equations
$a_{k}^{2}+d_{k}^{2}+2\left(b_{k}^{2}+c_{k}^{2}\right)=1, \quad k=1,2,3,4$,
ensuring the normalizations of the operators $\hat{U}^{(k)}$. The parameters $a_{1}, b_{1}, c_{1}, d_{1}$ in the operator $\hat{U}^{(1)}$ can be chosen arbitrarily, for example, $a_{1}=1, b_{1}=c_{1}=d_{1}=0$. Then from Eq. (25) we get $a_{2}=a_{3}=a_{4}=0$. Finally, there remain six equations from Eqs. (25) and (26) that can be used for determining the nine remaining unknown parameters. One can obtain three different solutions of these equations depending on how many of the parameters $d_{i}$ are allowed to be zero. In the following the corresponding operator sets are denoted by $\hat{U}_{1}^{(k)}, \hat{U}_{2}^{(k)}$ and $\hat{U}_{3}^{(k)}$, respectively. These solutions can be characterized by the number of freely chosen parameters.

Let us assume that
$d_{2} d_{3} d_{4} \neq 0$.
We introduce the notation
$b_{3} b_{4}+c_{3} c_{4}=\alpha_{2}, \quad b_{2} b_{4}+c_{2} c_{4}=\alpha_{3}, \quad b_{2} b_{3}+c_{2} c_{3}=\alpha_{4}$.
From Eqs. (25) and (27) it is clear that $\alpha_{i} \neq 0$ and
$\alpha_{2} \alpha_{3} \alpha_{4}=-\gamma^{2}<0$.
In this notation the solution is
$d_{2}^{2}=-\frac{2 \alpha_{3} \alpha_{4}}{\alpha_{2}}, \quad d_{3}^{2}=-\frac{2 \alpha_{2} \alpha_{4}}{\alpha_{3}}, \quad d_{4}^{2}=-\frac{2 \alpha_{3} \alpha_{2}}{\alpha_{4}}$,
and the corresponding operators $\hat{U}_{1}^{(k)}$ take the form

$$
\begin{array}{ll}
\hat{U}_{1}^{(1)}=\left(\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right) & \hat{U}_{1}^{(2)}=\left(\begin{array}{cc}
0 & b_{2}-\mathrm{i} c_{2} \\
b_{2}+\mathrm{i} c_{2} & \sqrt{2} \gamma / \alpha_{2}
\end{array}\right), \\
\hat{U}_{1}^{(3)}=\left(\begin{array}{cc}
0 & b_{3}-\mathrm{i} c_{3} \\
b_{3}+\mathrm{i} c_{3} & \sqrt{2} \gamma / \alpha_{3}
\end{array}\right), & \hat{U}_{1}^{(4)}=\left(\begin{array}{cc}
0 & b_{4}-\mathrm{i} c_{4} \\
b_{4}+\mathrm{i} c_{4} & \sqrt{2} \gamma / \alpha_{4}
\end{array}\right) . \tag{31}
\end{array}
$$

For the normalization equations (26) we get
$2 \gamma^{2} / \alpha_{k}^{2}+2 b_{k}^{2}+2 c_{k}^{2}=1, \quad k=2,3,4$.
These equations contain six parameters, hence, three of them can be chosen freely within certain constraints ensuring that the equations have real solutions satisfying the condition in Eq. (29).

The general form of the solutions of Eqs. (32) is complicated and depends on the three chosen independent parameters. Without presenting here these expressions explicitly, we present one set of dequantizer operators corresponding to the general form in Eq. (31):
$\hat{U}_{1}^{(1)}=\left(\begin{array}{ll}1 & 0 \\ 0 & 0\end{array}\right) \quad \hat{U}_{1}^{(2)}=\frac{1}{4}\left(\begin{array}{cc}0 & 2-\mathrm{i} \\ 2+\mathrm{i} & -\sqrt{6}\end{array}\right)$,
$\hat{U}_{1}^{(3)}=\frac{1}{4}\left(\begin{array}{cc}0 & 2+\mathrm{i} \\ 2-\mathrm{i} & \sqrt{6}\end{array}\right), \quad \hat{U}_{1}^{(4)}=\frac{1}{4}\left(\begin{array}{cc}0 & -\mathrm{i} \sqrt{6} \\ \mathrm{i} \sqrt{6} & 2\end{array}\right)$.
The other two solutions for the sets of dequantizers can be derived applying the conditions
$d_{2}=0, \quad d_{3} d_{4} \neq 0$
and
$d_{2}=d_{3}=0, \quad d_{4} \neq 0$.
Applying the procedure presented above at the first set of conditions one can derive the solution for the $\hat{U}_{2}^{(k)}$ operator set in the form
$\hat{U}_{2}^{(1)}=\left(\begin{array}{ll}1 & 0 \\ 0 & 0\end{array}\right), \quad \hat{U}_{2}^{(2)}=-\frac{c_{2}}{b_{4}}\left(\begin{array}{cc}0 & c_{4}+\mathrm{i} b_{4} \\ c_{4}-\mathrm{i} b_{4} & 0\end{array}\right)$,
$\hat{U}_{2}^{(3)}=\frac{c_{3}}{c_{4}}\left(\begin{array}{cc}0 & b_{4}-\mathrm{i} c_{4} \\ b_{4}+\mathrm{i} c_{4} & -\frac{2}{d_{4}}\left(b_{4}^{2}+c_{4}^{2}\right)\end{array}\right)$,
$\hat{U}_{2}^{(4)}=\left(\begin{array}{cc}0 & b_{4}-\mathrm{i} c_{4} \\ b_{4}+\mathrm{i} c_{4} & d_{4}\end{array}\right)$,
where
$b_{4}= \pm c_{2} \sqrt{\frac{2 c_{3}^{2}+2 c_{2}^{2}-1}{2 c_{2}^{2}-1}}$,
$c_{4}= \pm \sqrt{\frac{-2 c_{3}^{2}-2 c_{2}^{2}+1}{2}}$,
$d_{4}= \pm \frac{\sqrt{2} i c_{3}}{\sqrt{2 c_{2}^{2}-1}}$.

The latter expressions can be deduced from the corresponding normalization conditions. In this set the parameters $c_{2}$ and $c_{3}$ can be chosen freely within the restriction that all parameters are required to remain real $\left(\left|c_{2}\right| \leq 1 / \sqrt{2}\right.$ and $\left.\left|c_{3}\right| \leq 1 / \sqrt{2}\right)$.

At the conditions of Eq. (35) the corresponding operator set $\hat{U}_{3}^{(k)}$ can be expressed as

$$
\begin{align*}
& \hat{U}_{3}^{(1)}=\left(\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right), \quad \hat{U}_{3}^{(2)}=-\frac{c_{2}}{b_{3}}\left(\begin{array}{cc}
0 & c_{3}+\mathrm{i} b_{3} \\
c_{3}-\mathrm{i} b_{3} & 0
\end{array}\right), \\
& \hat{U}_{3}^{(3)}=\left(\begin{array}{cc}
0 & b_{3}-\mathrm{i} c_{3} \\
b_{3}+\mathrm{i} c_{3} & 0
\end{array}\right), \quad \hat{U}_{3}^{(4)}=\left(\begin{array}{cc}
0 & 0 \\
0 & d_{4}
\end{array}\right) \tag{40}
\end{align*}
$$

where
$c_{2}= \pm b_{3}$,
$c_{3}= \pm \frac{\sqrt{1-2 b_{3}^{2}}}{\sqrt{2}}$,
$d_{4}= \pm 1$.
In this case there is only one parameter denoted by $b_{3}$ that can be chosen freely within the restriction $\left|b_{3}\right| \leq 1 / \sqrt{2}$.

In Eqs. (31), (36), and (40) we presented the general form of minimal sets of dequantizers and quantizers for qubits containing three, two, and one parameters that can be chosen freely, respectively, within the restriction that all parameters have to remain real.

According to Eq. (17), any special minimal sets $\hat{V}^{(j)}$ of quantizers and dequantizers used for defining discrete Wigner functions can be derived from any type of the minimal sets presented in Eqs. (31), (36), and (40) by a linear transformation, so the connection can be expressed as
$\hat{V}^{(j)}=\sum_{k=1}^{4} L_{j k}^{(i)} \hat{U}_{i}^{(k)}, \quad j=1,2,3,4$.
Multiplying both sides by $\hat{U}_{i}^{\left(k^{\prime}\right)}$ and taking the trace of the expressions we get
$\operatorname{Tr}\left(\hat{V}^{(j)} \hat{U}_{i}^{\left(k^{\prime}\right)}\right)=\operatorname{Tr}\left(\sum_{k=1}^{4} L_{j k}^{(i)} \hat{U}_{i}^{(k)} \hat{U}_{i}^{\left(k^{\prime}\right)}\right)=\sum_{k=1}^{4} L_{j k}^{(i)} \operatorname{Tr}\left(\hat{U}_{i}^{(k)} \hat{U}_{i}^{\left(k^{\prime}\right)}\right)$.

Using Eq. (23) the expression on the right hand side simplifies to $L_{j k^{\prime}}^{(i)}$, therefore
$\operatorname{Tr}\left(\hat{V}^{(j)} \hat{U}_{i}^{\left(k^{\prime}\right)}\right)=L_{j k^{\prime}}^{(i)}$
In order to illustrate the application of the latter expression, let us consider the dequantizers

$$
\begin{array}{ll}
\hat{V}^{(1)}=\frac{1}{4}\left(\begin{array}{cc}
0 & 1+\mathrm{i} \\
1-\mathrm{i} & 2
\end{array}\right), & \hat{V}^{(2)}=\frac{1}{4}\left(\begin{array}{cc}
2 & 1-\mathrm{i} \\
1+\mathrm{i} & 0
\end{array}\right), \\
\hat{V}^{(3)}=\frac{1}{4}\left(\begin{array}{cc}
0 & -1-\mathrm{i} \\
-1+\mathrm{i} & 2
\end{array}\right), & \hat{V}^{(4)}=\frac{1}{4}\left(\begin{array}{cc}
2 & -1+\mathrm{i} \\
-1-\mathrm{i} & 0
\end{array}\right), \tag{45}
\end{array}
$$

presented in Ref. [44] for deriving a discrete Wigner function.
As an example, for the set of dequantizers of Eq. (40) with the choice of $b_{3}=1 / 2$, the linear transformation $L^{(3)}$ deduced from Eq. (44) takes the form
$L^{(3)}=\frac{1}{2}\left(\begin{array}{cccc}0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0\end{array}\right)$.
An interesting aspect of our results is that we found as many different types of minimal sets, i.e., three, as the number of mutually unbiased bases in this space. Recall that the different types of discrete Wigner functions that can be derived according to the approach introduced in [40] are associated with the mutually unbiased bases.

The generalization of the presented method of finding minimal sets of dequantizers and quantizers for higher dimensions is not obvious owing to the large number of parameters. In $d$ dimensions the number of operators forming the minimal set is $d^{2}$ containing $d^{4}$ parameters, while the number of equations following from the orthogonality and normalization conditions is $d^{2}\left(d^{2}+1\right) / 2$. Nonetheless, one can construct the minimal sets of dequantizers and quantizers for an N -qubit system using the tensor product of one-qubit dequantizers and quantizers.

## 5. Conclusion

We analyzed the general properties of minimal sets of dequantizers and quantizers for finite-dimensional quantum systems. We developed a general approach for deriving the corresponding quantizers assuming that a minimal set of dequantizers is known, and we have described the connection between different minimal sets. We have derived explicit expressions describing all minimal sets of dequantizers and quantizers for a qubit.

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