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On the information completeness of quantum tomograms

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Abstract

We address the problem of information completeness of quantum measurements in connection to quantum state tomography and with particular concern to quantum symplectic tomography. We put forward some non-trivial situations where informatively incomplete set of tomograms allows as well the state reconstruction provided to have some a priori information on the state or its dynamics. We then introduce a measure of information completeness and apply it to symplectic quantum tomograms.

1. Introduction

The problem of how to achieve a kind of measurement that is "complete" in the sense that it can be used to infer information on all possible (also exclusive) observables dates back to [1]. Obviously enough, no set of sharp observables can be informatively complete, while a set of (partially) non-commuting non-sharp observables can be informatively complete [2]. The problem of determining minimal sets of informatively complete observables is equivalent to the group theoretical problem of finding quantum tomographic schemes (still unsolved in its generality [3]). Following to the approach introduced in [4, 5] probability representations forming the quantum tomograms can be represented as scalar products of states with some elements of abstract Hilbert space. The information completeness can be checked from this point-of-view also.

Here we shall consider a quantum system on infinite dimensional Hilbert space $\mathcal{H} = L^2(\mathbb{R})$. Any measurement is characterized by a positive operator valued measure (POVM) $\hat{M}$ inferring a positive operator $\hat{M}(\Omega)$ to each Borel set $\Omega \subset \mathbb{R}$. Then, the result of measurement on a state $\hat{\rho}$ by means of POVM $\hat{M}$ is a probability distribution defined by $P(\Omega) = \mathrm{Tr}(\hat{\rho}\hat{M}(\Omega))$. The question that arises is: how much POVMs $\hat{M}^\epsilon$, labelled by the parameter ϵ, we should know to reconstruct the state $\hat{\rho}$ from a set of probability distributions $P^\epsilon(\Omega) = \mathrm{Tr}(\hat{\rho}\hat{M}^\epsilon(\Omega))$?

As we shall show a number of POVMs $\hat{M}^\epsilon$ which is informatively complete, in the sense that it allows the reconstruction of the state $\hat{\rho}$ from P^ϵ, depends on $\hat{\rho}$. If we use for $\hat{M}^\epsilon$ the orthogonal resolutions of the identity for the linear combinations of the position and momentum operators $\hat{X} = \mu\hat{x} + \nu\hat{p}$ [$\epsilon = (\mu, \nu)$ with $\mu, \nu \in \mathbb{R}$], then the densities $\omega(X, \mu, \nu)$ of probability distributions $P^\epsilon = P^{\mu,\nu}$ are called symplectic quantum tomograms [6]. By fixing a positive number r and putting $\mu = r\cos\theta$, $\nu = r\sin\theta$, it is known that the set of all rotated position distributions $\omega(X, \theta) = \omega(X, r\cos\theta, r\sin\theta)$ is informatively complete, because there is a one-to-one correspondence between the set of tomograms $\{\omega(X, \theta), \theta \in [0, \pi]\}$ and the Wigner function of the quantum state of the system [7]. This implies the need of an infinite number of tomograms to get information completeness. However, in practice these are never available. Hence, it would be important to identify situations where an incomplete set of symplectic tomograms allows as well the state reconstruction. It would be also important to quantify the information completeness of a set of tomograms.

Here we shall put forward non-trivial situations where an incomplete set of symplectic tomograms allows the state recon-

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struction, provided to have some a priori information on the state or its dynamics. Moreover, we shall introduce a measure of information completeness and we apply it to symplectic quantum tomograms.

2. State reconstruction by incomplete knowledge of symplectic tomograms

Let us define a two-parameter set of unitary transforms $\mathcal{F}_{\mu,\nu}$ in the space $\mathcal{H} = L^2(\mathbb{R})$ by the formula

$$(\mathcal{F}_{\mu,\nu}\psi)(x) = \frac{1}{\sqrt{2\pi|\nu|}} \int_{\mathbb{R}} e^{-i\frac{xy}{\nu}+i\frac{\mu y^2}{2\nu}} \psi(y)\,dy, \quad \nu \neq 0.$$

If $\psi \in \mathcal{H}$ is a wave function in the coordinate representation, then symplectic quantum tomogram $\omega(X,\mu,\nu)$ corresponding to the pure state $|\psi\rangle\langle\psi|$ can be written as

$$\omega(x,\mu,\nu) = \|\mathcal{F}_{\mu,\nu}\psi\|^2.$$

Ideally information completeness is achieved with an infinite number of the above tomograms. Nevertheless, in some cases a finite number of the above tomograms (incomplete knowledge of tomograms) might suffice for quantum state reconstruction, provided to have some a priori information about the state or its dynamics.

Below we consider some examples.

2.1. Finite number of nodes

Let us consider a particle moving in a one-dimensional potential $V(x)$. In the following we shall claim to know that

(a) the position probability distribution $\omega(t,X,1,0) \equiv \omega(X,1,0)$ has M nodes at the initial time t.

We shall call a state $|\psi\rangle\langle\psi|$ compatible with $\omega(X,1,0)$ if one has equalities $|\langle X|\psi(t)\rangle|^2 = \omega(t,X,1,0)$ and $|\langle X|\psi(t+\Delta t)\rangle|^2 = \omega(t+\Delta t,X,1,0)$ infinitesimally, i.e., for $\Delta t \to 0$.

We denote by $\mathcal{A}$ and the real numbers $-\infty = x_0 < x_1 < x_2 < \cdots < x_M < x_{M+1} = +\infty$ the set of possible states compatible with the distribution $\omega(X,1,0)$ and the nodes of $\omega(X,1,0)$, respectively. It was shown in [8] that the evolution equation for the tomogram results in the inclusion $|\psi\rangle\langle\psi|, |\xi\rangle\langle\xi| \in \mathcal{A}$ iff there exist phases $\phi_k \in [0,2\pi]$, $1 \leqslant k \leqslant M$, such that

$$\langle X|\xi\rangle = e^{i\phi_k}\langle X|\psi\rangle, \quad X \in [x_{k-1},x_k], \tag{1}$$

$1 \leqslant k \leqslant M+1$.

Now we suppose to know one more tomogram $\omega(X,\mu,\nu)$, $\nu \neq 0$. Then, we introduce the notation

$$\psi_{j,\mu,\nu} = \mathcal{F}_{\mu,\nu}(\chi_{[x_{j-1},x_j]}\psi),$$

where $\chi_{[x_{j-1},x_j]} = 1$ in the interval $[x_{j-1},x_j]$ and zero otherwise. Taking into account equality (1), we can write

$$\sum_{j\neq k} e^{i(\phi_k-\phi_j)}\psi_{k,\mu,\nu}(X)\psi^*_{j,\mu,\nu}(X) = \omega(X,\mu,\nu) - \sum_{j=1}^{M+1} \left|\psi_{j,\mu,\nu}(X)\right|^2. \tag{2}$$

This means that by knowing n tomograms $\omega(X,\mu_s,\nu_s)$, $1 \leqslant s \leqslant n$, we can write a system of equations of the form (2) for unknown phases ϕ_k, $1 \leqslant k \leqslant M+1$. It is shown in [8] that the matrix $(\psi_{j,\mu,\nu}(X)\psi_{k,\mu,\nu}(X)^*)_{j,k=1}^{M+1}$ is invertible if $n = M$, then there exists a unique solution to the system (2) and the information given by distributions $\omega(X,1,0)$, $\omega(X,\mu_s,\nu_s)$, $1 \leqslant s \leqslant n$, is complete.

2.2. Finite number of different phases

Let us suppose there exist a fragmentation $-\infty = x_{-n} < x_{-n+1} < \cdots < 0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = +\infty$ and a collection of numbers $0 \leqslant \phi_j < 2\pi$, $1 \leqslant j \leqslant n$, such that

$$\langle X|\psi\rangle = \sum_{j=-n+1}^{n} e^{i\phi_j}\chi_{[x_{j-1},x_j]}\psi_j(X),$$

where $\psi_j(X) \geqslant 0$, $X \in \operatorname{supp}\psi_j = [x_{j-1},x_j]$, $1 \leqslant j \leqslant n+1$. In the following we shall claim to know

(a) the fragmentation $\{x_j, -m \leqslant j \leqslant n\}$;
(b) the square of the wave function $|\psi(X)|^2 = \omega(X,1,0)$.

Our purpose is to study how can we reconstruct the state by means of the partial knowledge about tomograms $\omega(X,\mu,\nu) = \omega(X, r\cos\theta, r\sin\theta)$. We shall show that if a number of different phases equals $m+n$, we only need $m+n$ additional angles θ_j for which we know $\omega(X, r_j\cos\theta_j, r_j\sin\theta_j)$ to reconstruct the state.

First notice that the conditions (a) and (b) allow us to reconstruct the functions ψ_j as follows

$$\psi_j(X) = \chi_{[x_{j-1},x_j]}\sqrt{\omega(X,1,0)}.$$

Then, let us define a family of functions $\psi_{j,\mu,\nu}$ by the formula

$$\psi_{j,\mu,\nu} = \mathcal{F}_{\mu,\nu}\psi_j, \quad \nu \neq 0.$$

Since $\mathcal{F}_{\mu,\nu}$ is a unitary transformation, we get

$$\langle\psi_{j,\mu,\nu}|\psi_{k,\mu,\nu}\rangle = \langle\psi_j|\psi_k\rangle = \delta_{jk}.$$

Moreover,

$$\omega(X,\mu,\nu) = \sum_{j=-m+1}^{n} \left|\psi_{j,\mu,\nu}(X)\right|^2 + 2\sum_{j\neq k} \operatorname{Re}\left(e^{i(\phi_j-\phi_k)}\psi_{j,\mu,\nu}(X)\psi^*_{k,\mu,\nu}(X)\right). \tag{3}$$

Denoting $a_{jk\mu\nu}(X) = \operatorname{Re}(\psi_{j,\mu,\nu}\psi^*_{k,\mu,\nu})$ and $b_{jk\mu\nu}(X) = \operatorname{Im}(\psi_{j,\mu,\nu}\psi^*_{k,\mu,\nu})$, we can rewrite (3) as

$$\sum_{j\neq k}\left[a_{jk\mu\nu}(X)\cos(\phi_j-\phi_k)-b_{jk\mu\nu}(X)\sin(\phi_j-\phi_k)\right]$$
$$=\frac{1}{2}\left[\omega(X,\mu,\nu)-\sum_{j=-m+1}^{n}\left|\psi_{j,\mu,\nu}(X)\right|^2\right]. \tag{4}$$

The system of functions $\{f_{jk}(X)=\psi_{j,\mu,\nu}(X)\psi_{k,\mu,\nu}(X)^*, j\neq k\}$ can be linearly dependent. Thus, quite generally it would not be possible to solve Eq. (4) with respect to unknown phases. Note however that one can solve this equation if the time evolution of the system obeys some additional conditions (see, e.g., [9] and the discussion in the previous section).

Suppose that $|x_j-x_{j-1}|=\delta=\text{const}$ is sufficiently small and $x_j=\delta j$, then

$$\psi_j(X)=c_j+o(\delta),\quad \delta\to 0,\ x_{j-1}\leqslant X\leqslant x_j,\ 1\leqslant j\leqslant n,$$

where $c_j\geqslant 0$. In such a case we get

$$\psi_{j,0,\nu}(X)\approx\frac{2\sqrt{\nu}c_j}{\sqrt{2\pi}X}e^{-i\frac{X\delta j}{\nu}}e^{i\frac{\delta}{2}}\sin\frac{\delta}{2}$$

and, for $\mu\neq 0$,

$$\psi_{j,\mu,\nu}(X)\approx\frac{2\sqrt{\nu}c_j}{\sqrt{2\pi}X}e^{i\frac{\mu}{2\nu}(X-\delta j)^2}e^{-i\frac{\mu}{2\nu}X^2}e^{i\frac{\delta}{2}}\sin\frac{\delta}{2}.$$

Thus, the functions $f_{jk}=\psi_{j,\mu,\nu}\psi^*_{k,\mu,\nu}$ are linearly independent for different $j-k$. Moreover,

$$\psi_{j,\mu,\nu}\psi^*_{k,\mu,\nu}=e^{i\frac{\mu}{2\nu}\delta^2(j^2-k^2)}\psi_{j,0,\nu}\psi^*_{k,0,\nu}.$$

Notice that $j^2-k^2=(j-k)(j+k)$ and the system $j-k=r$, $j+k=s$ has a unique solution for the fixed pair (r,s). It follows that we can pick up $m+n$ angles $\theta_j\in[0,2\pi)$ and write out $m+n$ equations of the form (4) for $\mu=r_j\cos\theta_j$, $\nu=r_j\sin\theta_j$ such that solving this system we finally obtain the unknown phase differences $\phi_j-\phi_k$.

2.3. Free moving particle

The Schrödinger equation describing the motion of a free particle is defined as follows

$$i\psi_t=\frac{\hat{p}^2}{2}\psi. \tag{5}$$

The solution to (5) is given by the Fresnel integral

$$\psi(X,t)=\frac{1}{\sqrt{2\pi it}}\int_{\mathbb{R}}\exp\left(i\frac{(X-Y)^2}{2t}\right)\psi(0,Y)\,dY. \tag{6}$$

The probability representation of quantum states based upon the Fresnel tomography was introduced in [10–12]. Let us compare (6) with the Fresnel tomogram $\omega_F(X,\nu)$ (see formula (8) in [10]):

$$\omega_F(X,\nu)=\left|\frac{1}{\sqrt{2\pi i\nu}}\int_{\mathbb{R}}\exp\left(i\frac{(X-Y)^2}{2\nu}\right)\psi(0,Y)\,dY\right|^2.$$

It follows that if we know the symplectic quantum tomogram in the coordinate representation $\omega(t,X,1,0)=|\psi(X,t)|^2$ for all moments of time t, then we can reconstruct the Fresnel tomogram

$$\omega_F(X,\nu)=\omega(\nu,X,1,0).$$

It follows that the symplectic quantum tomogram in the initial moment of time $t=0$ can be derived from the square of wave function in the coordinate representation $|\psi(X,t)|^2$ known for all moments of time by the formula (compare with formula (9) in [10]):

$$\omega(0,X,\mu,\nu)=\frac{1}{|\mu|}\left|\psi\left(\frac{X}{\mu},\frac{\nu}{\mu}\right)\right|^2.$$

Thus, knowing the dynamics of the symplectic quantum tomogram only in the coordinate representation we can reconstruct the full tomogram in the initial moment of time.

2.4. Parametric driven oscillator

The dynamical problem of a parametric driven oscillator with frequency $\omega(t)$, force $f(t)$ depending on time and Hamiltonian

$$\hat{H}(t)=\frac{\hat{p}^2}{2}+\frac{\omega^2(t)\hat{x}^2}{2}-f(t)\hat{x} \tag{7}$$

was solved by the method of time-dependent linear in the position and momentum integrals of motion in [13,14].

In the following we shall claim to know

(a) $\omega(t)$, $f(t)$ and that the evolution takes place according to the Hamiltonian (7).

We denote by $\mathcal{M}(t,X,\mu,\nu)$ a one-parameter family of distribution functions associated with the dynamics of quantum tomograms $\omega_{\hat{\rho}(t)}(X,\mu,\nu)$ driven by the Hamiltonian (7) such that $i\hat{\rho}_t=[\hat{H},\hat{\rho}]$.

Let $\varepsilon(t)$, $\delta(t)$ be functions satisfying the equations $[\omega(0)=1]$

$$\ddot{\varepsilon}(t)+\omega^2(t)\varepsilon(t)=0,$$
$$\varepsilon(0)=1,\qquad \dot{\varepsilon}(0)=i,$$
$$\dot{\delta}(t)=-\frac{i}{\sqrt{2}}\varepsilon(t)f(t),\qquad \delta(0)=0.$$

Then, the dynamics of the distribution functions $\mathcal{M}(t,X,\mu,\nu)$ is given by the following formula [15]:

$$\mathcal{M}(t,X,\mu,\nu)$$
$$=\mathcal{M}\big(0,X+\sqrt{2}\,\mathrm{Re}\big((\mu\varepsilon+\nu\dot{\varepsilon})\delta^*\big),\mu\,\mathrm{Re}(\varepsilon)+\nu\,\mathrm{Re}(\dot{\varepsilon}),$$
$$\mu\,\mathrm{Im}(\varepsilon)+\nu\,\mathrm{Im}(\dot{\varepsilon})\big). \tag{8}$$

This means that if we know the evolution $\mathcal{M}(t,X,\mu,\nu)$ only for the values $\mu=1$ and $\nu=0$, then we can get all tomograms $\omega(X,\mu,\nu)$ at the initial moment $t=0$ by means of the formula (8) as follows

$$\omega\big(X,\mathrm{Re}(\varepsilon),\mathrm{Im}(\varepsilon)\big)=\frac{\partial}{\partial X}\big(\mathcal{M}\big(t,X-\sqrt{2}\,\mathrm{Re}\big(\varepsilon\delta^*\big),1,0\big)\big).$$

3. A measure of information completeness

In many situations no a priori information is available on the state, hence it would be helpful to have a measure of information completeness to use with quantum tomograms. To define such a measure we first need to define a measure on convex sets of states. To this end we exploit the informational measure introduced and investigated in [16].

3.1. The general case

Given a statistical ensemble $\{\pi_j, \hat{\rho}_j\}$ consisting of a probability distribution (π_j) on a set of states $(\hat{\rho}_j)$, one can consider the Holevo χ quantity [17]

$$\chi\big(\{\pi_j, \hat{\rho}_j\}\big) = S\left(\sum_j \pi_j \hat{\rho}_j\right) - \sum_j \pi_j S(\hat{\rho}_j),$$

where $S(\hat{\rho}) = -\operatorname{Tr}(\hat{\rho}\log\hat{\rho})$ is the von Neumann entropy. Let $\mathcal{A}$ be a convex set of states with finite von Neumann entropy. Then, the informational measure of $\mathcal{A}$ is defined by the formula [16]:

$$\bar{C}(\mathcal{A}) = \sup_{\{\pi_j, \hat{\rho}_j\}} \chi\big(\{\pi_j, \hat{\rho}_j\}\big),$$

where the supremum is taken over all probability distributions (π_j) on subsets of states $\hat{\rho}_j \in \mathcal{A}$. Notice that $\bar{C}(\mathcal{A})$ is monotonic with respect to $\mathcal{A}$, $\bar{C}(\mathcal{A}) < +\infty$ iff $\mathcal{A}$ is relatively compact and $\bar{C}(\mathcal{A}) = 0$ iff the set $\mathcal{A}$ consists of a single pure state (see [16, Theorem 2]).

Let us suppose that $\hat{M}^\epsilon$ is a set of POVMs labelled by the parameter ϵ and that the measurements of the unknown state $\hat{\rho}$ by means of $\hat{M}^\epsilon$ result in the set of probability distributions P^ϵ. Then, we consider the set $\mathcal{A}$ consisting of the states $\hat{\sigma}$ with the property

$$\operatorname{Tr}\big(\hat{\sigma}\hat{M}^\epsilon(\Omega)\big) = P^\epsilon(\Omega) \tag{9}$$

for all parameters ϵ and Borel sets $\Omega \subset \mathbb{R}$.

Now, the quantity $\bar{C}(\mathcal{A})$ can be considered as a measure of information completeness for the set $\hat{M}^\epsilon$. In particular, if $\mathcal{A}$ consists of only a single state, then $\bar{C}(\mathcal{A}) = 0$ and the set $\hat{M}^\epsilon$ is informatively complete.

3.2. Application to symplectic quantum tomograms

Let us suppose to know the position probability distribution $\omega(X, 1, 0)$. We denote by $\mathcal{A}$ the set of states generated by all wave functions resulting in the probability distribution $\omega(X, 1, 0)$. One can see that $|\psi\rangle\langle\psi|$, $|\phi\rangle\langle\phi| \in \mathcal{A}$ iff they are connected by the formula

$$\langle X|\psi\rangle = e^{i\xi(X)}\langle X|\phi\rangle, \quad X \in \operatorname{supp}\omega(X, 1, 0) \tag{10}$$

for some measurable function $\xi(X)$.

Moreover, we suppose to know N additional distributions $\omega(X, \mu_n, \nu_n)$, $\nu_n \neq 0$, $1 \leqslant n \leqslant N$. Let $\mathcal{A}$ be the set of states generated by all wave functions compatible with the distributions $\omega(X, 1, 0)$, $\omega(X, \mu_n, \nu_n)$, $1 \leqslant n \leqslant N$. In this case, we have N additional relations for $|\psi\rangle\langle\psi| \in \mathcal{A}$ of the following form,

$$\big|\mathcal{F}_{\mu_n,\nu_n}(\psi)(X)\big|^2 = \omega(X, \mu_n, \nu_n), \tag{11}$$

$1 \leqslant n \leqslant N$. These relations would decrease the set $\mathcal{A}$ leading in some limit cases to $\bar{C}(\mathcal{A}) = 0$.

Example *(Gaussian states).* A generic zero mean Gaussian state can be described by the characteristic function [17]

$$\Phi(x, y) = \exp\left[-\frac{1}{2}\big(\sigma_{xx}x^2 + 2\sigma_{xp}xy + \sigma_{pp}y^2\big)\right], \tag{12}$$

where $x, y \in \mathbb{R}$ and the covariances σ_{xx}, σ_{pp}, σ_{xp} satisfy the Schrödinger–Robertson uncertainty relation [14]

$$\sigma_{xx}\sigma_{pp} - \sigma_{xp}^2 \geqslant \frac{1}{4}.$$

The symplectic quantum tomograms corresponding to the characteristic function (12) are given by

$$\omega(X, \mu, \nu) = \frac{1}{\sqrt{2\pi}(\sigma_{xx}\mu^2 + 2\sigma_{xp}\mu\nu + \sigma_{pp}\nu^2)^{1/2}} \times \exp\left(-\frac{X^2}{2(\sigma_{xx}\mu^2 + 2\sigma_{xp}\mu\nu + \sigma_{pp}\nu^2)}\right).$$

Suppose to only know the tomogram

$$\omega(X, 1, 0) = \frac{1}{\sqrt{2\pi\sigma_{xx}}}\exp\left(-\frac{X^2}{2\sigma_{xx}}\right). \tag{13}$$

From it we can retrieve the covariance σ_{xx}. Let us calculate the measure $\bar{C}(\mathcal{A})$ for the set $\mathcal{A}$ consisting of the Gaussian states compatible with the distribution (13), i.e., with covariance σ_{xx}. This quantity equals to the maximum von Neumann entropy overall states in $\mathcal{A}$ [16].

In passing we note that the von Neumann entropy of a Gaussian state results [18]

$$S(\hat{\rho}) = g\left(\sqrt{\sigma_{xx}\sigma_{pp} - \sigma_{xp}^2} - \frac{1}{2}\right) \tag{14}$$

with

$$g(x) = (x+1)\log(x+1) - x\log x.$$

Then, because the condition (13) does not restrict the value of σ_{pp} we obtain for our case $\bar{C}(\mathcal{A}) = +\infty$.

Now suppose that besides the tomogram (13), we also know the tomogram

$$\omega(X, 0, 1) = \frac{1}{\sqrt{2\pi\sigma_{pp}}}\exp\left(-\frac{X^2}{2\sigma_{pp}}\right). \tag{15}$$

From it we can retrieve the covariance σ_{pp}. Then, taking into account (13), (15) and (14) we get

$$\bar{C}(\mathcal{A}) = g\left(\sqrt{\sigma_{xx}\sigma_{pp}} - \frac{1}{2}\right).$$

Finally, if we know any other tomogram for additional parameters $(\mu, \nu) \neq (1, 0)$ or $(0, 1)$, it will allow us to retrieve the covariance σ_{xp}. Since we supposed a priori that the set $\mathcal{A}$ is generated by pure states, we obtain that our Gaussian state is pure, i.e., $\sigma_{xp}^2 = \sigma_{xx}\sigma_{pp} - \frac{1}{4}$, so that $\bar{C}(\mathcal{A}) = 0$.

4. Conclusion

We have addressed the problem of informational completeness of quantum measurements in connection to quantum state tomography and with particular concern to quantum symplectic tomography. We have put forward some relevant cases where the state reconstruction is possible by incomplete knowledge of symplectic quantum tomograms. We have then introduced a measure of information completeness and we have applied it to symplectic quantum tomograms. This work sheds further light on the subject of quantum state characterization which is becoming relevant for many purposes, e.g., quantum information processing.

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