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Center of mass tomography for reconstructing quantum states of multipartite systems

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Abstract

We investigate a one random variable tomography map, describing quantum state of a multipartite system. The random variable is analogous to center of mass considered in rotated and scaled reference frames in the phase space. Starting from the star product formalism, we construct the map, investigate its properties (including the symmetry properties in respect to identical particles permutations), derive the evolution equation, and consider a multimode oscillator as an example. The physical meaning of the map is analyzed in detail.
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1. Introduction

During last decade there was a growing interest to new representation of quantum mechanics, a probability representation [1,2]. In the framework of this representation a quantum state is described by the *non-negative* probability distribution function, called marginal distribution [3,4], or tomogram (see also [5–7] for reference about the analogous quasidistribution functions: non-negative Husimi Q-function and Sudarshan–Glauber P-function).

Tomographic map, first developed for continuous variables, was then generalized for discrete spin variables [8–14]. It appears that the tomogram is a measurable quantity, that can be used in experiments on non-classical and coherent states of light or matter optics [15–26]. On the other hand, the non-negativity of state-describing

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function in tomographic representation attracts those who deal with the simulation of quantum systems [27]. This is due to the fact that many problems in the field arise because the values used to describe the state are can be negative, or even complex (for example, sign problem in Fermi-systems simulations). Tomography is also applicable in quantum computations and entanglement (see, e.g., [28]), as well as in the theory of information and signal analysis [20].

The state of the system with N degrees of freedom can be described by density matrix $\rho(\vec{q}', \vec{q}'')$ [29,30], the function of $2N$ variables. Usual symplectic tomography scheme [31–33] introduces the non-negative function of $3N$ variables, symplectic tomogram $w_s(\vec{X}, \vec{\mu}, \vec{\nu})$, to describe the state. Homogeneity of this function [34,35] is the reason that effectively it has only $2N$ independent variables. In connection with the growing interest to the tomography in experiments and simulations, it would be desirable to find more simple tomographic map with less number of variables. This is the aim of present Letter. We analyze in details the 'center-of-mass' tomography, described in Ref. [36], which operates with one random variable X. Advantage of this scheme is that the state describing function, $w(X, \vec{\mu}, \vec{\nu})$, depends on only $2N + 1$ variables.

In Section 2 we present the definition of tomographic scheme with one random variable, elucidate some of its useful properties and discuss the physical meaning of the map. In Section 3 we derive the equations describing quantum evolution, stationary states, quantum transitions and rules for average values calculation for the proposed tomography map. Some examples of state description using the developed approach are given in Section 4 and symmetry of the map with respect to particles permutations is discussed in Section 5. The work is summarized in Section 6.

2. One random variable tomography

2.1. Definition of the tomographic map

We begin with the one-dimensional (1D) case of a particle with continuous degree of freedom (in this Letter we do not consider spin variables, but generalization of the formalism is straightforward). Quantum mechanics states that we know 'everything' about the system if we know density matrix. In practice, to obtain any information about the system we have to measure some quantities, for example, coordinate q or momentum p. It is also possible sometimes to measure an intermediate quantity, $\mu q + \nu p$, where μ, ν are real parameters. Formally, this quantity (denote it X) is *coordinate, measured in scaled and rotated reference frame in the phase space*. It turns out, that the distribution function of X ($w(X, \mu, \nu)$), measured for all sets of μ, ν gives *complete quantum mechanical description* of the system, in the sense that there is a unique correspondence between $w(X, \mu, \nu)$ and density matrix (see, e.g., [1,31,33,37,38]). Note that distribution function $w(X, \mu, \nu)$ is equal to $\langle \delta(X - \mu\hat{q} - \nu\hat{p}) \rangle$, where $\langle \cdots \rangle$ is quantum mechanical average. Then there is, in principle, a possibility of complete experimental density matrix determination through the set of coordinate measurements.

When we deal with more than one particle and dimension we can consider individual $X_j = \mu_j q_j + \nu_j p_j$ for every jth degree of freedom. This results in the symplectic tomography representation [33,37]. Here we are to show that it is enough to work with only one $X = \sum_j X_j$. To do this, let us consider the generalization of $w(X, \mu, \nu) = \langle \delta(X - \mu\hat{q} - \nu\hat{p}) \rangle$, where q, p and μ, ν becomes the vectors, their products become scalar products of vectors, while X remains a real number:

$$w(X, \vec{\mu}, \vec{\nu}) = \left\langle \delta\left(X - \vec{\mu}\hat{\vec{q}} - \vec{\nu}\hat{\vec{p}}\right) \right\rangle. \tag{1}$$

Related problems were discussed in [21,22].

Throughout the Letter designations are the following. We consider the system of N particles in d dimensions, the number of degrees of freedom is Nd. Vectors are written as $\vec{a}$, we use everywhere the vectors with Nd components, if the otherwise is not mentioned. Designation $\vec{e}$ is used for the vector with all components equal to 1 ($e_j = 1$). Scalar product of vectors is designated $a = \vec{b}\vec{c}$ (meaning $a = \sum_j b_j c_j$), $\vec{a} = \vec{b} \circ \vec{c}$ denotes the component-wise

product of vectors ($a_j = b_j c_j$). The tomogram for usual symplectic scheme is designated as $w_s(\vec{X}, \vec{\mu}, \vec{\nu})$ ($\vec{X}, \vec{\mu}$ and $\vec{\nu}$ with Nd components each); the tomogram with one random variable is written as $w(X, \vec{\mu}, \vec{\nu})$. We also use Planck constant $\hbar = 1$ everywhere.

We can begin the construction of one-random-variable tomography representation from the known star product expressions. In the framework of star product formalism [39–42] every operator is replaced by the function ('symbol'), depending on specific set of parameters (y), products of operators turn into 'star products' of corresponding symbols (general star product problem was discussed also in Ref. [43] and symplectic tomography was considered from the point of view of star product in Ref. [33]). As a result, one deals with functions only, avoiding operators. For example, using a pair of operators $\hat{D}(y), \hat{U}(y)$, we construct the connection between the symbols $f_A(y)$ and operators $\hat{A}$:

$$f_A(y) = \mathrm{Tr}\big(\hat{A}\hat{U}(y)\big), \tag{2}$$

$$\hat{A} = \int f_A(y)\hat{D}(y)\,dy, \tag{3}$$

$$\int \mathrm{Tr}\big(\hat{D}(y)\hat{U}(y)\big)\,dy = 1. \tag{4}$$

For $y = \{X, \vec{\mu}, \vec{\nu}\}$ one can choose

$$\hat{U}(y) = \delta\big(X - \vec{\mu}\hat{\vec{q}} - \vec{\nu}\hat{\vec{p}}\big), \tag{5}$$

$$\hat{D}(y) = (2\pi)^{-Nd} \exp\big[i\big(X - \vec{\mu}\hat{\vec{q}} - \vec{\nu}\hat{\vec{p}}\big)\big], \tag{6}$$

which defines the symbols (denote them $w_A(X, \vec{\mu}, \vec{\nu})$) and star product:

$$(w_A * w_B)(y) = \int w_A(y'')w_B(y')K(y'', y', y)\,dy''\,dy'. \tag{7}$$

The kernel of star product $K(y'', y', y)$ is expressed as follows:

$$\begin{aligned} K(y'', y', y) &= \mathrm{Tr}\big[\hat{D}(y'')\hat{D}(y')\hat{U}(y)\big] \\ &= \int e^{-i(kX - X' - X'')}\delta(\vec{\mu}'' + \vec{\mu}' - k\vec{\mu})\delta(\vec{\nu}'' + \vec{\nu}' - k\vec{\nu}) \\ &\quad \times e^{-i(\vec{\mu}''\vec{\nu}' - k(\vec{\mu}''\vec{\nu}' + \vec{\mu}'\vec{\nu}) + (\vec{\mu}'\vec{\nu}' + \vec{\mu}''\vec{\nu}'' + k^2\vec{\mu}\vec{\nu})/2)}\frac{dk}{(2\pi)^{Nd+1}}. \end{aligned} \tag{8}$$

For any operator $\hat{A}$ we have $\langle A\rangle = \mathrm{Tr}(\hat{\rho}\hat{A})$, therefore, w_ρ-symbol of density operator $\hat{\rho}$ is the same as $w(X, \vec{\mu}, \vec{\nu})$ defined by Eq. (1). Density matrix in any representation is just a matrix element of density operator. Then, finally, we come to the unique correspondence (invertable map) between the tomogram w and density matrix mentioned above:

$$w(X, \vec{\mu}, \vec{\nu}) = \big\langle \delta\big(X - \vec{\mu}\hat{\vec{q}} - \vec{\nu}\hat{\vec{p}}\big)\big\rangle, \tag{9}$$

$$\hat{\rho} = \int w(X, \vec{\mu}, \vec{\nu})e^{i(X - \vec{\mu}\hat{\vec{q}} - \vec{\nu}\hat{\vec{p}})}\frac{dX\,d\vec{\mu}\,d\vec{\nu}}{(2\pi)^{Nd}}. \tag{10}$$

Density matrix always can be reconstructed from the tomogram w using these equations, so one random variable tomogram describes quantum state *completely*. Note that now the state-describing function is *nonnegative* and depends on $2Nd + 1$ variables in contrast to symplectic tomogram, depending on $3Nd$ variables.

2.2. Properties of $w(X, \vec{\mu}, \vec{\nu})$

It is convenient to consider density matrix in coordinate representation and Wigner description of quantum states [44] to derive the properties and evolution equation for the tomogram w. In the framework of Wigner formalism the state of the system is described by real Wigner function $W(\vec{q}, \vec{p})$ defined in phase space and connected with the density matrix as follows:

$$W(\vec{q}, \vec{p}) = \int \rho\left(\vec{q} + \frac{\vec{u}}{2}, \vec{q} - \frac{\vec{u}}{2}\right) e^{-i\vec{p}\vec{u}} \frac{d\vec{u}}{(2\pi)^{Nd}}, \tag{11}$$

$$\rho(\vec{q}', \vec{q}'') = \int W\left(\frac{\vec{q}' + \vec{q}''}{2}, \vec{p}\right) e^{i\vec{p}(\vec{q}' - \vec{q}'')}\, d\vec{p}. \tag{12}$$

Using Eqs. (9), (10) and Eqs. (11), (12) we obtain:

$$w(X, \vec{\mu}, \vec{\nu}) = \int W(\vec{q}, \vec{p}) e^{-ik(X - \vec{\mu}\vec{q} - \vec{\nu}\vec{p})} \frac{dk\, d\vec{q}\, d\vec{p}}{(2\pi)}, \tag{13}$$

$$W(\vec{q}, \vec{p}) = \int e^{-i(\vec{\mu}\vec{q} + \vec{\nu}\vec{p} - X)} w(X, \vec{\mu}, \vec{\nu}) \frac{dX\, d\vec{\mu}\, d\vec{\nu}}{(2\pi)^{2Nd}}. \tag{14}$$

Usual symplectic tomography map is developed in references [1,33,37]. The symplectic tomogram $w_s(\vec{X}, \vec{\mu}, \vec{\nu})$ and Wigner function are connected as follows:

$$w_s(\vec{X}, \vec{\mu}, \vec{\nu}) = \int W(\vec{q}, \vec{p}) e^{-i\vec{k}(\vec{X} - \vec{\mu}\circ\vec{q} - \vec{\nu}\circ\vec{p})} \frac{d\vec{k}\, d\vec{q}\, d\vec{p}}{(2\pi)^{Nd}}, \tag{15}$$

$$W(\vec{q}, \vec{p}) = \int e^{-i\vec{e}(\vec{\mu}\circ\vec{q} + \vec{\nu}\circ\vec{p} - \vec{X})} w_s(\vec{X}, \vec{\mu}, \vec{\nu}) \frac{d\vec{X}\, d\vec{\mu}\, d\vec{\nu}}{(2\pi)^{2Nd}}. \tag{16}$$

Since the Wigner function is connected by invertable maps with both tomograms w and w_s it is obvious that they contain the same information about the quantum state. In fact one has

$$w(X, \vec{\mu}, \vec{\nu}) = \int w_s(\vec{Y}, \vec{\mu}, \vec{\nu}) \delta\left(X - \sum_{j=1}^{Nd} Y_j\right) d\vec{Y}, \tag{17}$$

$$w_s(\vec{X}, \vec{\mu}, \vec{\nu}) = \int w(Y, \vec{k}\circ\vec{\mu}, \vec{k}\circ\vec{\nu}) e^{i(Y - \vec{k}\vec{X})}\, d\vec{k}\, dY. \tag{18}$$

The Wigner function is normalized:

$$\int W(\vec{q}, \vec{p})\, d\vec{q}\, d\vec{p} = \int \rho\left(\vec{q} + \frac{\vec{u}}{2}, \vec{q} - \frac{\vec{u}}{2}\right) e^{-i\vec{p}\vec{u}} \frac{d\vec{u}\, d\vec{q}\, d\vec{p}}{(2\pi)^{Nd}} = \int \rho\left(\vec{q} + \frac{\vec{u}}{2}, \vec{q} - \frac{\vec{u}}{2}\right) \delta(\vec{u})\, d\vec{u}\, d\vec{q}$$
$$= \int \rho(\vec{q}, \vec{q})\, d\vec{q} = 1, \tag{19}$$

where we choose the normalization for density matrix $\mathrm{Tr}(\hat{\rho}) = 1$. Then the tomogram w is normalized in X variable:

$$\int w(X, \vec{\mu}, \vec{\nu})\, dX = \int W(\vec{q}, \vec{p}) \delta(k) e^{ik(\vec{\mu}\vec{q} + \vec{\nu}\vec{p})}\, dk\, d\vec{q}\, d\vec{p} = 1. \tag{20}$$

Although the tomogram depends on $2Nd + 1$ variables, instead of $2Nd$ for density matrix, the completeness of physical description is the same for both formulations, due to the fact that the tomogram is a homogeneous

function. Consider the definition (13) and multiply all variables in w by a real number λ:

$$w(\lambda X, \lambda\vec{\mu}, \lambda\vec{\nu}) = \int W(\vec{q}, \vec{p}) e^{-i\lambda k(X-\vec{\mu}\vec{q}-\vec{\nu}\vec{p})} \frac{dk\, d\vec{q}\, d\vec{p}}{(2\pi)} = \int W(\vec{q}, \vec{p}) e^{-ik(X-\vec{\mu}\vec{q}-\vec{\nu}\vec{p})} \frac{dk\, d\vec{q}\, d\vec{p}}{(2\pi|\lambda|)}$$
$$= \frac{w(X, \vec{\mu}, \vec{\nu})}{|\lambda|}, \tag{21}$$

where we just made the change of variables $\lambda k \to k$.

From property (21) we have

$$w(X, \vec{\mu}, \vec{\nu}) = |X|^{-1} w(1, \vec{\mu}/X, \vec{\nu}/X). \tag{22}$$

A pure state is described by the wave function $\Psi(\vec{q})$. In this case w is given by

$$w(X, \vec{\mu}, \vec{\nu}) = \int d\vec{Y} \frac{\delta\left(X - \sum_{j=1}^{Nd} Y_j\right)}{(2\pi)^{Nd} \prod_{j=1}^{Nd} |\nu_j|} \left| \int \Psi(\vec{q}) \exp\left[i \left(\vec{q}\, \frac{\vec{Y}}{\vec{\nu}} - \frac{\vec{q} \circ \vec{q}}{2} \frac{\vec{Y}}{\vec{\nu}} \right) \right] d\vec{q} \right|^2. \tag{23}$$

Corresponding formula for the symplectic tomogram w_s was given in Ref. [45].

2.3. Physical meaning

We have defined the non-negative function $w(X, \vec{\mu}, \vec{\nu})$ (9) completely describing quantum state. For any set of $\{\vec{\mu}, \vec{\nu}\}$ it is normalized as a function of X, therefore, $w(X, \vec{\mu}, \vec{\nu})$ is the set of distribution functions of quantity X. Then to know the quantum state completely one has to consider all sets of $\{\vec{\mu}, \vec{\nu}\}$ (in practice, moving with some step) and measure $X = \vec{\mu}\vec{q} + \vec{\nu}\vec{p}$ many times for each set: this yields the distribution function $w(X, \vec{\mu}, \vec{\nu})$ for given set of $\{\vec{\mu}, \vec{\nu}\}$.

Looking at Eq. (22) we see that we even do not have to know $w(X, \vec{\mu}, \vec{\nu})$, the value of this function in some point in X for all $\{\vec{\mu}, \vec{\nu}\}$ is enough. This does not change the scheme of measurements, we still need to measure full distribution function of X for given $\{\vec{\mu}, \vec{\nu}\}$ (it is necessary to compare the values of distribution function in different points to be sure that statistical precision is good), but one has to store the smaller arrays of information.

Property (21) can be used in another way. If λ is equal to $\vec{\mu}\vec{\mu} + \vec{\nu}\vec{\nu}$, we can parameterize $\{\vec{\mu}, \vec{\nu}\}$ by λ and $2Nd - 1$ angles (to use the spherical coordinates in the space of $\{\vec{\mu}, \vec{\nu}\}$). Applying Eq. (21) we come to reduced tomogram with $2Nd$ variables and $\{\vec{\mu}, \vec{\nu}\}$ located on the sphere with radius equal unity in $(2Nd)$-dimensional space. This new tomogram also completely describes the state and in some cases it can be convenient to use this one in measurements, because it is easier to sample $2Nd - 1$ angles than $(2Nd)$-dimensional space from $-\infty$ to ∞ (see, e.g., [20,22]). On the other hand, such formulation causes trouble with the derivation of evolution equations and arbitrary average values calculation.

The only remaining unclear point is the meaning of $X = \vec{\mu}\vec{q} + \vec{\nu}\vec{p}$. It is the sum of positions measured in scaled and rotated reference frame in the phase space. But what does it mean physically? It is impossible to measure $\vec{q}$ and $\vec{p}$ simultaneously, but sometimes one can transform $\vec{q}$ and $\vec{p}$ into the form $\vec{\mu}\vec{q} + \vec{\nu}\vec{p}$, for example, mixing the signal beam with local oscillator field (in quantum optics, see [19] and references therein). Another scheme was proposed in [20], where $\vec{q}$ and $\vec{p}$ are mixed due to wave (electromagnetic or matter) propagation through a lens (or an analog of a lens in atomic optics). Taking into account the present development of science concerning controlling the Bose-condensates of atoms, this also can be a possible realm of tomography measurements. Bose-condensate is a coherent macroscopic state of many atoms and it is described by macroscopic wave function. For example, one can mix two such waves (condensates of the same atoms), using the first as a signal wave and the second as local oscillator. Varying the phase difference of the condensates we sample different $\vec{\mu}, \vec{\nu}$. Probably, the same can be done in superconductors (where the electrons of superconductivity also form the coherent macroscopic matter wave), using Josephson junctions.

If we somehow accomplished the scaling and rotation of reference frame in the phase space we can measure the set of positions in this reference frame $X_j = \mu_j q_j + \nu_j p_j$, but it is enough to measure their sum, $X = \vec{\mu}\vec{q} + \vec{\nu}\vec{p}$. It is analogous to the position of center of mass measurement (the sum of coordinates of corresponding vector, to be more precise). Indeed, the center of mass position is

$$X_{\mathrm{cm}} = \sum_j m_j X_j / M = \sum_j m_j (\mu_j q_j + \nu_j p_j)/M, \tag{24}$$

where $M = \sum_j m_j$ and m_j is the mass corresponding to jth degree of freedom, and X_{cm} can be associated with $X = \vec{\mu}\vec{q} + \vec{\nu}\vec{p}$ for some other set of $\{\vec{\mu}, \vec{\nu}\}$. We sample all sets of $\{\vec{\mu}, \vec{\nu}\}$, therefore, it is enough to measure the center-of-mass position in each scaled and rotated reference frame.

Finally, we would like to make the following remark. The storage of arrays representing full density matrix or tomogram becomes impossible when the number of degrees of freedom growth. If we use some grid, the number of arrays elements is proportional to n^{Nd}, where n is the number of grid steps. Increasing Nd we soon come to the situation when all data carriers in the world cannot store corresponding arrays. And this is not necessary as the state of the system is uniquely determined by the one-particle density (through the density functional, see [46] and references therein). Then for many-particles systems description we can use reduced density matrices (one-body, two-body, etc.), and tomography map is constructed for them in the same way as for full density matrix. Then the situation with reference frame scaling and rotation is simplified because μ and ν are the same for all particles (if one-body density matrix is considered) and distribution functions are averaged over all particles.

3. State transformations

3.1. Evolution equations

Let us discuss the evolution equation for tomogram w. Begin with the most general evolution equation for density matrix:

$$i\frac{\partial \rho(\vec{q}', \vec{q}'')}{\partial t} = \big[\hat{H}, \rho(\vec{q}', \vec{q}'')\big]. \tag{25}$$

Here and throughout the Letter we omit the dependence on time t, but imply that all functions, describing the state (density matrix, Wigner function, tomogram) depend on time as parameter. For Hamiltonians in the form $\hat{H} = \sum_i \hat{p}_i^2/(2m_i) + V(\vec{q})$ we can utilize the Moyal evolution equation for Wigner function [47,48]:

$$\frac{\partial W}{\partial t} + \frac{\vec{p}}{m}\frac{\partial W}{\partial \vec{q}} + i\left[V\left(\vec{q} + \frac{i}{2}\frac{\partial}{\partial \vec{p}}\right) - V\left(\vec{q} - \frac{i}{2}\frac{\partial}{\partial \vec{p}}\right)\right]W = 0, \tag{26}$$

where $\vec{p}/\vec{m}$ means the vector with components p_i/m_i (the equation holds for the case of different masses for different particles and directions), the operators in the potential V designates the analytical expansion of the potential and use of the products of corresponding operators. This equation can be easily obtained applying the transform (11) to Eq. (25).

Let us apply the transform (13) to evolution equation (26). Expanding the potential in Eq. (26), we come to the transforms of the following quantities: $\vec{q}W$, $\partial W/\partial \vec{q}$, $\vec{p}W$ and $\partial W/\partial \vec{p}$. The transform (13) of $\vec{q}W$ is

$$\begin{aligned}&\int \vec{q}\, W(\vec{q}, \vec{p}) \exp\big[-ik(X - \vec{\mu}\vec{q} - \vec{\nu}\vec{p})\big]\frac{dk\, d\vec{q}\, d\vec{p}}{(2\pi)} \\ &\quad = -i\frac{\partial}{\partial \vec{\mu}} \int \frac{W(\vec{q}, \vec{p})}{k} \exp\big[-ik(X - \vec{\mu}\vec{q} - \vec{\nu}\vec{p})\big]\frac{dk\, d\vec{q}\, d\vec{p}}{(2\pi)}.\end{aligned} \tag{27}$$

Using an operator $(\partial/\partial X)^{-1}$, giving the antiderivative of the function it works on, we have

$$i\frac{e^{-ikX}}{k} = \frac{i}{k}\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial X}e^{-ikX} = \left(\frac{\partial}{\partial X}\right)^{-1}e^{-ikX}, \tag{28}$$

and Eq. (27) becomes

$$\int \vec{q}\,W(\vec{q},\vec{p})\exp\bigl[-ik(X-\vec{\mu}\vec{q}-\vec{\nu}\vec{p})\bigr]\frac{dk\,d\vec{q}\,d\vec{p}}{(2\pi)} = -\frac{\partial}{\partial\vec{\mu}}\left(\frac{\partial}{\partial X}\right)^{-1}w(X,\vec{\mu},\vec{\nu}). \tag{29}$$

Analogously one can obtain the rules of transformation of the terms in Eq. (26), designated as '$\to$':

$$\vec{q}\,W(\vec{q},\vec{p}) \to -\frac{\partial}{\partial\vec{\mu}}\left(\frac{\partial}{\partial X}\right)^{-1}w(X,\vec{\mu},\vec{\nu}), \tag{30}$$

$$\frac{\partial W(\vec{q},\vec{p})}{\partial\vec{q}} \to \vec{\mu}\frac{\partial}{\partial X}w(X,\vec{\mu},\vec{\nu}), \tag{31}$$

$$\vec{p}\,W(\vec{q},\vec{p}) \to -\frac{\partial}{\partial\vec{\nu}}\left(\frac{\partial}{\partial X}\right)^{-1}w(X,\vec{\mu},\vec{\nu}), \tag{32}$$

$$\frac{\partial W(\vec{q},\vec{p})}{\partial\vec{p}} \to \vec{\nu}\frac{\partial}{\partial X}w(X,\vec{\mu},\vec{\nu}). \tag{33}$$

Application of Eqs. (30)–(33) to the Eq. (26) gives the evolution equation for one random variable quantum tomogram w:

$$\frac{\partial w}{\partial t} - \frac{\vec{\mu}}{m}\frac{\partial w}{\partial\vec{\nu}} + i\left[V\left(-\frac{\partial}{\partial\vec{\mu}}\left(\frac{\partial}{\partial X}\right)^{-1}+\frac{i}{2}\vec{\nu}\frac{\partial}{\partial X}\right) - V\left(-\frac{\partial}{\partial\vec{\mu}}\left(\frac{\partial}{\partial X}\right)^{-1}-\frac{i}{2}\vec{\nu}\frac{\partial}{\partial X}\right)\right]w = 0. \tag{34}$$

3.2. Stationary states and quantum transitions

Stationary states with definite energy obey the following eigenvalue equation:

$$\hat{H}\hat{\rho}_E = \hat{\rho}_E\hat{H} = E\hat{\rho}_E. \tag{35}$$

From Eq. (11) we have the rules of transition from the equation for density matrix to equation for Wigner function:

$$\frac{\partial^2\rho(\vec{q},\vec{q}')}{\partial\vec{q}^2} \to \left(\frac{1}{4}\frac{\partial^2}{\partial\vec{q}^2} - \vec{p}^2 + i\vec{p}\frac{\partial}{\partial\vec{q}}\right)W(\vec{q},p), \qquad V(\vec{q})\rho(\vec{q},\vec{q}') \to V\left(\vec{q}+\frac{i}{2}\frac{\partial}{\partial\vec{p}}\right)W(\vec{q},p). \tag{36}$$

Using (30)–(33), we have the eigenvalue equation for the tomogram w with one random variable:

$$\sum_{j=1}^{Nd}\left[\frac{1}{2m_j}\frac{\partial^2}{\partial\nu_j^2}\left(\frac{\partial}{\partial X}\right)^{-2} - \frac{1}{8m_j}\mu_j^2\frac{\partial^2}{\partial X^2}\right]w + \operatorname{Re}V\left(\frac{i}{2}\vec{\nu}\frac{\partial}{\partial X} - \frac{\partial}{\partial\vec{\mu}}\left(\frac{\partial}{\partial X}\right)^{-1}\right)w = Ew,$$

$$-\sum_{j=1}^{Nd}\frac{\mu_j}{2m_j}\frac{\partial w}{\partial\nu_j} = \operatorname{Im}V\left(\frac{i}{2}\vec{\nu}\frac{\partial}{\partial X} - \frac{\partial}{\partial\vec{\mu}}\left(\frac{\partial}{\partial X}\right)^{-1}\right)w. \tag{37}$$

Consider now two states, a and b. The probability of transition from state a to state b is $P_{ab} = \operatorname{Tr}(\hat{\rho}_a\hat{\rho}_b) = \int \rho_a(\vec{q}',q'')\rho_b(\vec{q}'',q')\,d\vec{q}'\,d\vec{q}''$, or, in terms of the Wigner formalism,

$$P_{ab} = (2\pi)^{Nd}\int W^a(\vec{q},\vec{p})W^b(\vec{q},\vec{p})\,d\vec{q}\,d\vec{p}. \tag{38}$$

From Eq. (14) one gets the expression for P_{ab} in tomography approach:

$$P_{ab} = \int w^a(X, \vec{\mu}, \vec{\nu}) w^b(Y, -\vec{\mu}, -\vec{\nu}) e^{i(X+Y)} \frac{dX\, dY\, d\vec{\mu}\, d\vec{\nu}}{(2\pi)^{Nd}}. \tag{39}$$

3.3. Tomographic map in temperature-dependent processes

Tomography map can be as well applied to the systems in thermodynamic equilibrium, with temperature $T \neq 0$. Tomographic evolution equation in 'imaginary time' $\beta = 1/T$ (measuring T in units of energy) is derived from the following equation:

$$-\frac{\partial \rho(\vec{q}', \vec{q}'', \beta)}{\partial \beta} = \hat{H}_{\vec{q}'} \rho(\vec{q}', \vec{q}'', \beta), \tag{40}$$

with initial condition $\rho(\vec{q}', q'', \beta = 0) = \delta(\vec{q}' - q'')$ (corresponding to constant Wigner function, see Eq. (11)). Here index $\vec{q}'$ in $\hat{H}_{\vec{q}'}$ shows that the Hamiltonian acts only on variables $\vec{q}'$.

Transition to the tomogram w is straightforward. We just use the same rules, as in the derivation of evolution equation (34) and eigenvalue equation (37). Then the evolution equation in imaginary time β for w is given by

$$\begin{aligned} -\frac{\partial w}{\partial \beta} &= \sum_{j=1}^{Nd} \left[\frac{1}{2m_j} \frac{\partial^2}{\partial \nu_j^2} \left(\frac{\partial}{\partial X} \right)^{-2} - \frac{1}{8m_j} \mu_j^2 \frac{\partial^2}{\partial X^2} \right] w + \operatorname{Re} V \left(\frac{i\vec{\nu}}{2} \frac{\partial}{\partial X} - \frac{\partial}{\partial \vec{\mu}} \left(\frac{\partial}{\partial X} \right)^{-1} \right) w, \\ -\sum_{j=1}^{Nd} \frac{\mu_j}{2m_j} \frac{\partial w}{\partial \nu_j} &= \operatorname{Im} V \left(\frac{i}{2} \vec{\nu} \frac{\partial}{\partial X} - \frac{\partial}{\partial \vec{\mu}} \left(\frac{\partial}{\partial X} \right)^{-1} \right) w. \end{aligned} \tag{41}$$

From Eq. (13) we see, that initial condition ($\beta = 0$) is w in the delta-function form, equal zero everywhere, besides the point $\vec{\mu}$, $\nu = 0$ and constant in X direction in that point.

3.4. Average values calculation

Developing the 'center-of-mass' tomography formalism we must provide the rules of average values calculation to complete the picture. Using the density matrix to describe the state of the system we can obtain the average value of some operator $\hat{A}$ as

$$\langle A \rangle = \operatorname{Tr}(\hat{\rho} \hat{A}), \tag{42}$$

where we choose $\operatorname{Tr}(\hat{\rho}) = 1$.

In the framework of Wigner–Moyal formulation of quantum mechanics one deals with the *Weyl symbol* $A^W(\vec{q}, \vec{p})$ [49] of operator $A(\hat{\vec{q}}, \hat{\vec{p}})$ (see [50,51] for review, see also [52]), to calculate the average value:

$$\langle A \rangle = \int A^W(\vec{q}, \vec{p}) W(\vec{q}, \vec{p})\, d\vec{q}\, d\vec{p}, \tag{43}$$

where the Weyl symbol is given by

$$A^W(\vec{q}, \vec{p}) = \int \operatorname{Tr}\left(A(\hat{\vec{q}}, \hat{\vec{p}}) e^{i\vec{\xi}\hat{q} + i\vec{\eta}\hat{p}} \right) e^{-i\vec{\xi}q - i\vec{\eta}p} \frac{d\vec{\xi}\, d\vec{\eta}}{(2\pi)^{2Nd}}. \tag{44}$$

Expression for the average values in one random variable tomography formulation is obtained using the connection between w and Wigner function (14):

$$\langle A \rangle = \int e^{iX} w(X, \vec{\mu}, \vec{\nu}) A(\vec{\mu}, \vec{\nu})\, dX\, d\vec{\mu}\, d\vec{\nu}, \tag{45}$$

$$A(\vec{\mu}, \vec{\nu}) = \int A^W(\vec{q}, \vec{p}) e^{-i(\vec{\mu}\vec{q} + \vec{\nu}\vec{p})} \frac{d\vec{q}\, d\vec{p}}{(2\pi)^{2Nd}}. \tag{46}$$

If considered operator depends on coordinates $\hat{\vec{q}}$ or momenta $\hat{\vec{p}}$ only, Weyl symbols have the same form as corresponding operators in coordinate or momentum representation. Operator $A(\hat{\vec{q}})$ is $A(\vec{x})$ in $\vec{x}$-coordinate representation, then its Weyl symbol $A^W(\vec{q}, p)$ is equal to $A(\vec{q})$. The same is valid for momenta-dependent operator: $B(\hat{\vec{p}})$ is $B(\vec{y})$ in $\vec{y}$-momentum representation, and $B^W(\vec{q}, p) = B(\vec{p})$.

Consider an operator $A(\hat{\vec{q}})$, depending on coordinates only. For momenta-dependent operators all equations are the same, provided μ is replaced by ν, and vice versa, because the pairs $\vec{q}, \vec{\mu}$ and $\vec{p}, \vec{\nu}$ enter the equations connecting the tomogram w with Wigner function symmetrically. Integration over $\vec{\nu}$ in Eq. (45) for operator $A(\hat{\vec{q}})$ gives the delta-function $\delta(\vec{\nu})$. Then we have:

$$\langle A\rangle = \int A^W(\vec{q}) e^{-i(\vec{\mu}q - X)} w(X, \vec{\mu}, \vec{\nu} = 0) \frac{dX\, d\vec{\mu}\, d\vec{q}}{(2\pi)^{Nd}}. \tag{47}$$

It is often necessary to operate with the one-particle and one-dimension operators. Then, quite generally, we can consider an operator $A(\hat{q}_1)$. Corresponding average value is given by

$$\langle A\rangle = \int A^W(X) w\big(X, \mu_1 = 1, \tilde{\vec{\mu}} = 0, 0\big)\, dX, \tag{48}$$

where $\tilde{\vec{\mu}}$ designates all μ_j except the specified μ_1.

4. Examples

In this section we introduce several examples of tomographic map for many-particles quantum states. For simplicity, here we do not regard symmetry over particles exchange. Permutations properties are considered in Section 5.

4.1. Gaussian states

In various applications we often deal with the pure states and wave functions of Gaussian form. Examples are the ground state, as well as coherent or squeezed states of the system of non-interacting oscillators, or some many-dimensional Gaussian wave packet. Such wave packet can be created due to parametric excitation of multimode vacuum state of electromagnetic field [53], e.g., in the framework of nonstationary Casimir effect [54].

For the wave function $\Psi(\vec{q}) = \prod_{j=1}^{Nd} \psi_j(q_j)$, where

$$\psi_j(q) = (A_j/\pi)^{1/4} e^{-\frac{A_j}{2}(q-x_j)^2 + iy_j q}, \tag{49}$$

we have (using Eq. (11)) the Wigner function as a product of $W_j(q_j, p_j)$, where

$$W_j(q, p) = e^{-A_j(q-x_j)^2} e^{-B_j(p-y_j)^2} (A_j B_j)^{1/2}/\pi, \tag{50}$$

and for states (49) $B_j = 1/A_j$. Here we used the fact that the Fourier transform of a Gaussian is Gaussian. The same rule works when we apply the transformation (13) to Eq. (50). Then, the tomogram of Gaussian states has the form

$$w^{\text{Gauss}}(X, \vec{\mu}, \vec{\nu}) = \frac{e^{-(X - \vec{\mu}\vec{x} - \vec{\nu}\vec{y})^2/C}}{\sqrt{\pi C}}, \qquad C = \sum_{j=1}^{Nd} \left(\frac{\mu_j^2}{A_j} + \frac{\nu_j^2}{B_j}\right). \tag{51}$$

Thermal density matrix of independent oscillators is also Gaussian, but it is not a product of wave functions, as the state is not pure. Still it is a product of density matrices of individual oscillators (see, e.g., [55]):

$$\rho_j(q, q') = \sqrt{\frac{2A_j(B_j - 1)}{\pi}}\, e^{-A_j[B_j(q^2 + q'^2) - 2qq']}, \tag{52}$$

where $A_j = m\omega_j/(2\,\mathrm{sh}(\omega_j\beta))$ and $B = \mathrm{ch}(\omega_j\beta)$. Omitting the straightforward calculations, we obtain the tomogram w in the following form:

$$w^{(\beta)}(X,\vec{\mu},\vec{\nu}) = \frac{e^{-X^2/D}}{\sqrt{\pi D}}, \qquad D = \sum_{j=1}^{Nd}\left(\frac{\mu_j^2}{2A_j(B_j-1)} + 2\nu_j^2 A_j(B_j+1)\right). \tag{53}$$

4.2. Fock states

The Fock states of light (the eigenstates in representation of photons number) correspond to ground or excited states of multimode oscillator. The state is labeled by vector $\vec{n}$ of integer numbers and wave function has the form:

$$\Psi(\vec{q}) = \prod_{j=1}^{Nd}\frac{e^{-q_j^2/2}H_{n_j}(q_j)}{\pi^{1/4}\sqrt{2^{n_j}n_j!}}, \tag{54}$$

where H_m is the Hermit polynomial of mth order. To obtain the tomogram for such state we use the following facts. First, coherent state of an oscillator is described by the Gaussian wave function and, correspondingly, by the Gaussian tomogram (see Eq. (51)). Coherent state is labeled by complex vector $\vec{\alpha} = \vec{a} + i\vec{b}$ and parameters of Gaussian wave function in coordinate representation (49) are $x_j = \sqrt{2}\,a_j$ and $y_j = -\sqrt{2}\,b_j$. Second, the wave function of coherent state (for simplicity, one dimension is considered here) is expanded in the basis of Fock states as

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{55}$$

which is connected with the expression for generating function of Hermit polynomials:

$$e^{-\alpha^2+2\alpha q} = \sum_{n=0}^{\infty}\frac{\alpha^n}{n!}H_n(q). \tag{56}$$

Expanding the tomogram of coherent state in Hermit polynomials and wave function of coherent state in corresponding integral expression in wave functions of Fock states we have

$$w^{\vec{n}}(X,\vec{\mu},\vec{\nu}) = \int \delta\left(X - \sum_{j=1}^{Nd}X_j\right)\prod_{j=1}^{Nd}\frac{H_{n_j}^2\left(X_j/\sqrt{\mu_j^2+\nu_j^2}\right)\exp\left\{-X_j^2/(\mu_j^2+\nu_j^2)\right\}}{2^{n_j}n_j!\sqrt{\pi(\mu_j^2+\nu_j^2)}}\,d\vec{X}. \tag{57}$$

For example, for $N = 2, d = 1$ and states with n_1, n_2 equal to 0 or 1 (denoted (n_1, n_2)) the tomograms $w(X,\mu_1,\mu_2,\nu_1,\nu_2)$ have the forms

$$w^{(0,0)} = \frac{\exp[-X^2/C]}{\sqrt{\pi C}}, \tag{58}$$

$$w^{(0,1)} = \sqrt{\frac{C_2}{\pi C_1}}\frac{(2C_2X^2 + C_1C_2 + C_1^2)e^{-X^2/C}}{C^{5/2}}, \tag{59}$$

$$w^{(1,1)} = \frac{4C_1^2C_2^2e^{-X^2/C}}{\sqrt{\pi}\,C^{5/2}}\left(\frac{X^4}{C^2} + \frac{X^2}{C}\frac{C_1^2+C_2^2-4C_1C_2}{C_1C_2} + \frac{3}{4}\right), \tag{60}$$

where $C_1 = \mu_1^2 + \nu_1^2$, $C_2 = \mu_2^2 + \nu_2^2$ and $C = C_1 + C_2$.

5. Symmetry properties with respect to particles permutations

Consideration of identical particles exchange imposes the restrictions concerning the possible form of the state-describing functions. In this section we discuss the corresponding properties of one-random-variable tomographic map (see [56] for permutation symmetry properties of the symplectic tomogram).

Further we use the following notations. A vector without index $\vec{a}$ has Nd components, vector with index $\vec{a}_j$ denotes the set of some values, corresponding to jth particle, and consists of d components. A vector $\tilde{\vec{a}}$ denotes the collection of all components of $\vec{a}$, except those that are specified in the same expression. For example, $\tilde{\vec{q}}$ in the expression $\psi(\vec{q}_j, \tilde{\vec{q}})$ is the vector of all the coordinates, except the coordinates of the jth particle.

For particles obeying Fermi or Bose statistics, we have the following symmetry properties concerning their permutations:

$$\rho(\vec{q}'_j, \vec{q}'_i, \tilde{\vec{q}}'; \vec{q}''_i, \vec{q}''_j, \tilde{\vec{q}}'') = \rho(\vec{q}'_i, \vec{q}'_j, \tilde{\vec{q}}'; \vec{q}''_j, \vec{q}''_i, \tilde{\vec{q}}'') = \pm\rho(\vec{q}'_i, \vec{q}'_j, \tilde{\vec{q}}'; \vec{q}''_i, \vec{q}''_j, \tilde{\vec{q}}''), \tag{61}$$

where the upper sign ('+') is for Bose systems, and lower sign ('−') is for Fermi systems. Note that 'entire' particles permutation (two particles exchange both q and q' variables) corresponds to *sign conservation for both Fermi and Bose statistics*:

$$\rho(\vec{q}'_j, \vec{q}'_i, \tilde{\vec{q}}'; \vec{q}''_j, \vec{q}''_i, \tilde{\vec{q}}'') = \rho(\vec{q}'_i, \vec{q}'_j, \tilde{\vec{q}}'; \vec{q}''_i, \vec{q}''_j, \tilde{\vec{q}}''). \tag{62}$$

In the expressions for obtaining the Wigner function form density matrix (11) and tomogram w from Wigner function (13) we can exchange the integration variables ($\vec{u}_j \leftrightarrow \vec{u}_i$, etc.), then we immediately have:

$$W(\vec{q}_j, \vec{q}_i, \tilde{\vec{q}}; \vec{p}_j, \vec{p}_i, \tilde{\vec{p}}) = W(\vec{q}_i, \vec{q}_j, \tilde{\vec{q}}; \vec{p}_i, \vec{p}_j, \tilde{\vec{p}}), \tag{63}$$

$$w(X; \vec{\mu}_j, \vec{\mu}_i, \tilde{\vec{\mu}}; \vec{\nu}_j, \vec{\nu}_i, \tilde{\vec{\nu}}) = w(X; \vec{\mu}_i, \vec{\mu}_j, \tilde{\vec{\mu}}; \vec{\nu}_i, \vec{\nu}_j, \tilde{\vec{\nu}}). \tag{64}$$

We see that there is no distinction between Fermi and Bose statistics when the particles exchange 'entirely', i.e. q and q' in the density matrix, q and p in Wigner function or μ, ν in w are permuted *simultaneously*. The distinction appears when not all the variables, corresponding to the considered particles, are permuted. When we use the density matrix, Fermi and Bose statistics differ only in the sign ± 1, which appears after the permutation of either $\vec{q}'_i, \vec{q}'_j$ or $\vec{q}''_i, \vec{q}''_j$. For the Wigner function and tomogram this difference is expressed in far more complicated manner, through the integral transforms (see corresponding formulae for the symplectic tomography in [56]).

First, regard the permutation of $\vec{q}_i, \vec{q}_j$ or $\vec{p}_i, \vec{p}_j$ for the Wigner function. Again exchanging the integration variables in (11) we come to

$$W(\vec{q}_j, \vec{q}_i, \tilde{\vec{q}}; \vec{p}_i, \vec{p}_j, \tilde{\vec{p}}) = W(\vec{q}_i, \vec{q}_j, \tilde{\vec{q}}; \vec{p}_j, \vec{p}_i, \tilde{\vec{p}}). \tag{65}$$

The same considerations lead us to the similar expression for w:

$$w(X; \vec{\mu}_j, \vec{\mu}_i, \tilde{\vec{\mu}}; \vec{\nu}_i, \vec{\nu}_j, \tilde{\vec{\nu}}) = w(X; \vec{\mu}_i, \vec{\mu}_j, \tilde{\vec{\mu}}; \vec{\nu}_j, \vec{\nu}_i, \tilde{\vec{\nu}}). \tag{66}$$

Then it is enough to develop the formulae for coordinate (Wigner function) or $\vec{\mu}$ (tomogram) permutations only. Corresponding integral expressions has the following form:

$$W(\vec{q}_j, \vec{q}_i, \tilde{\vec{q}}; \vec{p}_i, \vec{p}_j, \tilde{\vec{p}}) = \int K^W(\vec{x}_i, \vec{x}_j, \vec{y}_i, \vec{y}_j, \vec{q}_i, \vec{q}_j, \vec{p}_i, \vec{p}_j) W(\vec{x}_i, \vec{x}_j, \tilde{\vec{q}}; \vec{y}_i, \vec{y}_j, \tilde{\vec{p}})\, d\vec{x}_i\, d\vec{x}_j\, d\vec{y}_i\, d\vec{y}_j, \tag{67}$$

$$w(X, \vec{\mu}_j, \vec{\mu}_i, \tilde{\vec{\mu}}, \vec{\nu}_i, \vec{\nu}_j, \tilde{\vec{\nu}}) = \int K(X, \vec{\mu}_i, \vec{\mu}_j, \vec{\nu}_i, \vec{\nu}_j; Y, \vec{\xi}_i, \vec{\xi}_j, \vec{\eta}_i, \vec{\eta}_j) w(Y, \vec{\xi}_i, \vec{\xi}_j, \tilde{\vec{\mu}}, \vec{\eta}_i, \vec{\eta}_j, \tilde{\vec{\nu}})\, dY\, d\vec{\xi}_i\, d\vec{\xi}_j\, d\vec{\eta}_i\, d\vec{\eta}_j, \tag{68}$$

and kernels are given by

$$K^W(\vec{x}_i,\vec{x}_j,\vec{y}_i,\vec{y}_j,\vec{q}_i,\vec{q}_j,\vec{p}_i,\vec{p}_j) = \pm\left(\frac{4}{2\pi}\right)^d \delta(\vec{x}_i+\vec{x}_j-\vec{q}_i-\vec{q}_j)\delta(\vec{y}_i+\vec{y}_j-\vec{p}_i-\vec{p}_j)e^{i[(\vec{q}_i-\vec{q}_j)(\vec{y}_i-\vec{y}_j)+(\vec{x}_i-\vec{x}_j)(\vec{p}_i-\vec{p}_j)]}, \tag{69}$$

$$K(X,\vec{\mu}_i,\vec{\mu}_j,\vec{\nu}_i,\vec{\nu}_j;Y,\vec{\xi}_i,\vec{\xi}_j,\vec{\eta}_i,\vec{\eta}_j) = \pm\int \frac{|k|^{2d}}{(2\pi)^{d+1}}\delta(\vec{\xi}_i+\vec{\xi}_j-\vec{\mu}_i-\vec{\mu}_j)\delta(\vec{\eta}_i+\vec{\eta}_j-\vec{\nu}_i-\vec{\nu}_j) \times e^{-i\{k(X-Y)-k^2/4[(\vec{\mu}_i-\vec{\mu}_j)(\vec{\eta}_i-\vec{\eta}_j)+(\vec{\xi}_i-\vec{\xi}_j)(\vec{\nu}_i-\vec{\nu}_j)]\}}\,dk. \tag{70}$$

6. Conclusion

The tomographic scheme for which the quantum state of multipartite system is associated with a probability distribution function of *one random variable* was analyzed in details. Specific set of parameters $\vec{\mu}$ and $\vec{\nu}$ determines certain scaling and rotation of the reference frame in the phase space, while the position of the system center of mass, measured in this reference frame, corresponds to the random variable X. The center of mass tomogram is shown to contain the same information about the state as the symplectic tomogram, Wigner function or density matrix in the coordinate or momentum representations do. The expressions connecting the center of mass tomogram to these state-describing functions are given in explicit form. The construction of the symplectic tomography relation to quantum probability measure theory [2] can be easily extended to the center of mass tomography approach too.

Quantum evolution equations and energy level equations for the introduced center-of-mass tomogram were developed. Example of multimode oscillator, symmetry properties of the tomogram for identical particles (fermions and bosons) and physical meaning of the map were discussed in detail.

The suggested center of mass tomography scheme can be used as an additional tool for reconstructing the density matrix of states of multipartite systems, alternatively to known tomographic schemes, such as optical and symplectic ones.

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