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Quantum probability measure for parametric oscillators

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Abstract

A driven oscillator with time-dependent frequency is studied in the framework of quantum probability measure approach. Propagator for the measure evolution of the parametric oscillator is obtained in explicit form. Relation to tomographic description of the parametric oscillator states is established.

1. Introduction

Recently [1] the quantum probability measure approach [2] was related to tomographic description of quantum states [3–6] and their evolution. The aim of this work is to study in detail the quantum probability measure approach to the problem of parametric excitation of the harmonic oscillator with time-dependent parameters. The driven oscillator with time-dependent frequency was considered in [7]. The driven oscillator and its excitation were studied also in [8,9]. In addition to known solutions of the Schrödinger equation for the parametric oscillator of both types like Gaussian packets (squeezed states) and Fock states obtained in [7] the integrals of motion were found in [10] (quadratic Ermakov type invariants) and in [11] (linear in position and momentum). The problem of parametric oscillator models the behaviour of ion in Paul trap (see, e.g., [12]) and it was studied in tomographic probability picture in [13]. We will study and solve the evolution equation for the quantum probability measure related to homodyne quadrature for the driven parametric oscillator with time-dependent frequency. Propagator for the evolution equation for the quantum probability measure will be found in the framework of the approach discussed.

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2. Probability measures associated with quantum evolution

Suppose that the quantum evolution is described by the equation

$$\partial_t \hat{\rho} = -i[\widehat{H}, \hat{\rho}], \tag{1}$$

where $\hat{\rho}$ is a quantum state and $\widehat{H}$ is an arbitrary Hamiltonian. Taking position and momentum operators $\hat{x}$ and $\hat{p}$ let us involve a spectral decomposition of the operator $\mu\hat{x} + \nu\hat{p}$:

$$\mu\hat{x} + \nu\hat{p} = \int_{\mathbb{R}} X \, d\widehat{M}. \tag{2}$$

Here $\widehat{M}$ is an orthogonal positive operator-valued measure on the real line $\mathbb{R}$. Then given a Borel subset $\Omega \subset \mathbb{R}$ the formula [2]

$$\mathcal{M}_{\hat{\rho}}(\Omega) = \mathrm{Tr}\big(\hat{\rho}\widehat{M}(\Omega)\big) \tag{3}$$

defines a probability measure $\mathcal{M}_{\hat{\rho}}$ on the real line depending on the solution $\hat{\rho}$ to Eq. (1). The distribution function $\mathcal{M}(t, \mu, \nu, X) = \mathcal{M}_{\hat{\rho}}((-\infty, X])$ of the measure (3) is an integral functional from the solution $\hat{\rho}$ to Eq. (1) which can be represented in the form [1]

$$\mathcal{M}(t, \mu, \nu, X) = \frac{1}{2\pi|\nu|} \int_{-\infty}^{X} dx \int_{\mathbb{R}} dy \int_{\mathbb{R}} dy' \, \rho(y, y') \exp\left(i\frac{(y^2 - y'^2)\mu}{2\nu} - i\frac{x(y - y')}{\nu}\right). \tag{4}$$

The distribution function (4) is connected with the quantum tomogram $\omega_\rho(t, \mu, \nu, X)$ of the state ρ by the formula (see [1])

$$\omega_\rho(t, \mu, \nu, X) = \frac{d}{dX}\mathcal{M}(t, \mu, \nu, X). \tag{5}$$

If $\rho = |\psi\rangle\langle\psi|$ is a pure state, the formula (4) is transformed to

$$\mathcal{M}(t, \mu, \nu, X) = \frac{1}{2\pi|\nu|} \int_{-\infty}^{X} dx \left| \int_{\mathbb{R}} dy \, \psi(y) \exp\left(i\frac{\mu y^2}{2\nu} - i\frac{xy}{\nu}\right)\right|^2. \tag{6}$$

In [1] it was derived the evolution equation for the functional $\mathcal{M}(t, \nu, \mu, X)$ for the case of Hamiltonian $\widehat{H} = \frac{\hat{p}^2}{2} + V(\hat{x})$:

$$\begin{aligned} &\frac{\partial\mathcal{M}(t, \mu, \nu, X)}{\partial t} - \mu\frac{\partial\mathcal{M}(t, \mu, \nu, X)}{\partial \nu} \\ &\quad - i\left(V\left(-\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\mu} - \frac{i\nu}{2}\frac{\partial}{\partial X}\right) - V\left(-\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\mu} + \frac{i\nu}{2}\frac{\partial}{\partial X}\right)\right)\mathcal{M}(t, \mu, \nu, X) = 0. \end{aligned}$$

3. Excited and coherent states of driven oscillator

A driven oscillator with time-dependent force is described by the Hamiltonian

$$\widehat{H}(t) = \frac{\hat{p}^2}{2} + \frac{\omega(t)^2\hat{x}^2}{2} - f(t)\hat{x}. \tag{7}$$

The integrals of motion for the driven oscillator have the form [14],

$$\hat{A}(t) = \frac{i}{\sqrt{2}}\big(\epsilon(t)\hat{p} - \dot{\epsilon}(t)\hat{x}\big) + \delta(t),$$
$$\hat{A}^{\dagger}(t) = -\frac{i}{\sqrt{2}}\big(\epsilon^*(t)\hat{p} - \dot{\epsilon}^*(t)\hat{x}\big) + \delta^*(t),$$

where the functions $\epsilon(t)$ and $\delta(t)$ satisfy the equations

$$\ddot{\epsilon}(t) + \omega^2(t)\epsilon(t) = 0,$$
$$\epsilon(0) = 1, \qquad \dot{\epsilon}(0) = i,$$
$$\dot{\delta}(t) = -\frac{i}{\sqrt{2}}\epsilon(t) f(t), \qquad \delta(0) = 0.$$

The wave function ψ_0 of a ground state of driven oscillator is determined by the condition $\hat{A}\psi_0 = 0$ which gives us the formula

$$\psi_0(t,x) = \frac{1}{\pi^{1/4}|\epsilon|^{1/2}} \exp\left(\frac{i\dot{\epsilon}}{2\epsilon}x^2 - \frac{\sqrt{2}\delta}{\epsilon}x - \frac{\delta^2\epsilon^*}{2\epsilon} - \frac{|\delta|^2}{2} + \frac{1}{2}\int_0^t \big(\dot{\delta}\delta^* - \delta\dot{\delta}^*\big)\,d\tau\right).$$

Notice that to obtain the last formula it needs to use the Schrödinger equation for an unknown part of ψ_0 which does not depend on x. Let us introduce the unitary operator $\widehat{\mathcal{D}}(\alpha) = \exp(\alpha\hat{A}^{\dagger} - \alpha^*\hat{A})$, then the wave functions $\psi_\alpha(t,x)$ of coherent states can be written as

$$\psi_\alpha(t,x) = \big(\widehat{\mathcal{D}}(\alpha)\psi_0\big)(t,x) \exp\left(\frac{\sqrt{2}\alpha x}{\epsilon} - \frac{\alpha^2\epsilon^*}{2\epsilon} - \frac{|\alpha|^2}{2} + \frac{\alpha(\delta\epsilon^* + \delta^*\epsilon)}{\epsilon}\right)\psi_0(t,x). \tag{8}$$

Then one can calculate the wave functions $\psi_n(t,x)$ of excited states as

$$\psi_n(t,x) = \big((\hat{A}^{\dagger})^n\psi_0\big)(t,x) = \left(\frac{\epsilon^*}{2\epsilon}\right)^{n/2}\frac{1}{n!}H_n\left(\frac{x + \frac{1}{\sqrt{2}}(\delta\epsilon^* + \delta^*\epsilon)}{|\epsilon|}\right)\psi_0(t,x). \tag{9}$$

4. Evolution of probability measures associated with driven oscillator

The Hamiltonian (7) of driven oscillator acts on operators $\mu\hat{x} + \nu\hat{p}$ by the formula

$$[\widehat{H}, \mu\hat{x} + \nu\hat{p}] = \frac{\mu}{2}[\hat{p}^2, \hat{x}] + \frac{\nu\omega^2}{2}[\hat{x}^2, \hat{p}] - \nu f[\hat{x}, \hat{p}] = -i\mu\hat{p} + i\nu\omega^2\hat{x} - i\nu f \equiv \hat{\Lambda}(\mu,\nu).$$

Notice that the operator $\hat{\Lambda}(\mu,\nu)$ commutes with the operator $\mu\hat{x} + \nu\hat{p}$. It follows that

$$\big[\widehat{H}, F(\mu\hat{x} + \nu\hat{p})\big] = F'(\mu\hat{x} + \nu\hat{p})\hat{\Lambda}(\mu,\nu)$$

for an arbitrary differentiable function F. A value $M(\Omega)$ of the spectral operator-valued measure (2) is equal to a characteristic function $\chi_\Omega(\mu\hat{x} + \nu\hat{p})$, where $\chi_\Omega(x) = 1$, $x \in \Omega$, $\chi_\Omega(x) = 0$, $x \notin \Omega$. Hence for the distribution function $\mathcal{M}(t,\mu,\nu,X)$ of the probability measure $\mathcal{M}_{\hat{\rho}}$ defined by (3) we get

$$\begin{aligned}\frac{\partial\mathcal{M}(t,\mu,\nu,X)}{\partial t} &= \operatorname{Tr}\frac{\partial\hat{\rho}}{\partial t}M\big((-\infty,X]\big) = -i\operatorname{Tr}[\widehat{H},\hat{\rho}]M\big((-\infty,X]\big)\\ &= i\operatorname{Tr}\hat{\rho}\big[\widehat{H}, M\big((-\infty,X]\big)\big] = i\operatorname{Tr}\hat{\rho}\delta(X - \mu\hat{x} - \nu\hat{p})\hat{\Lambda}(\mu,\nu)\\ &= \mu\frac{\partial\mathcal{M}(t,\mu,\nu,X)}{\partial\nu} - \nu\omega^2\frac{\partial\mathcal{M}(t,\mu,\nu,X)}{\partial\mu} + \nu f\frac{\partial\mathcal{M}(t,\mu,\nu,X)}{\partial X}.\end{aligned}$$

Thus we have derived the evolution equation for the distribution function of the probability measure associated with driven oscillator:

$$\frac{\partial \mathcal{M}(t,\mu,\nu,X)}{\partial t} = \mu \frac{\partial \mathcal{M}(t,\mu,\nu,X)}{\partial \nu} - \nu\omega^2 \frac{\partial \mathcal{M}(t,\mu,\nu,X)}{\partial \mu} + \nu f \frac{\partial \mathcal{M}(t,\mu,\nu,X)}{\partial X}. \tag{10}$$

It is straightforward to check that the integrals of motion for Eq. (10) are

$$\begin{aligned}
&I_1 = \mu\epsilon + \nu\dot{\epsilon}, \qquad I_2 = \mu\epsilon^* + \nu\dot{\epsilon}^* \equiv I_1^*,\\
&I_3 = \mu^2|\epsilon|^2 + \nu^2|\dot{\epsilon}|^2 + 2\mu\nu\,\mathrm{Re}(\epsilon^*\dot{\epsilon}) \equiv I_1 I_2,\\
&I_4 = X + \sqrt{2}\big(\mu\,\mathrm{Re}(\epsilon\delta^*) + \nu\,\mathrm{Re}(\dot{\epsilon}\delta^*)\big) \equiv X + \frac{1}{\sqrt{2}} I_1\delta^* + \frac{1}{\sqrt{2}} I_2\delta.
\end{aligned}$$

Deducing μ, ν and X from the expressions for integrals under the condition $t = 0$, we get

$$\mu = \frac{1}{2}(I_1 + I_2), \qquad \nu = \frac{1}{2}(I_1 - I_2), \qquad X = I_4.$$

It follows that the propagator $T(t)$ giving a solution to Eq. (10) acts by the formula

$$\begin{aligned}
\mathcal{M}(t,\mu,\nu,X) &= T(t)\big(\mathcal{M}(0,\mu,\nu,X)\big) = \mathcal{M}\big(0, \tfrac{1}{2}(I_1+I_2), \tfrac{1}{2}(I_1-I_2), I_4\big)\\
&= \mathcal{M}\Big(0, \mu\,\mathrm{Re}(\epsilon) + \nu\,\mathrm{Re}(\dot{\epsilon}), \mu\,\mathrm{Im}(\epsilon) + \nu\,\mathrm{Im}(\dot{\epsilon}), X + \sqrt{2}\big(\mu\,\mathrm{Re}(\epsilon\delta^*) + \nu\,\mathrm{Re}(\dot{\epsilon}\delta^*)\big)\Big).
\end{aligned} \tag{11}$$

At the initial time $t = 0$ for the probability measure of oscillator $\mathcal{M}(0,\mu,\nu,X)$ coincides with the distribution function of stationary oscillator calculated in [1]. Then the time-dependent solution is given by the formula (11). In particular, for the wave function of coherent state (8) we obtain

$$\mathcal{M}_\alpha(t,\mu,\nu,X) = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{X + \sqrt{2}(\mu\,\mathrm{Re}((\delta-\alpha)\epsilon^*) + \nu\,\mathrm{Re}((\delta-\alpha)\dot{\epsilon}^*))}{\sqrt{\mu^2|\epsilon|^2 + \nu^2|\dot{\epsilon}|^2 + 2\mu\nu\,\mathrm{Re}(\epsilon\dot{\epsilon}^*)}}\right)\right). \tag{12}$$

Analogously, for probability measures of excited states (9) we get

$$\mathcal{M}_n(t,\mu,\nu,X) = \frac{1}{n!}\frac{d^n}{dt^n}\frac{d^n}{ds^n}\left(\frac{1}{2}e^{ts}\left(1 + \mathrm{erf}\left(\frac{X + \sqrt{2}(\mu\,\mathrm{Re}(\delta\epsilon^*) + \nu\,\mathrm{Re}(\delta\dot{\epsilon}^*))}{\sqrt{\mu^2|\epsilon|^2 + \nu^2|\dot{\epsilon}|^2 + 2\mu\nu\,\mathrm{Re}(\epsilon\dot{\epsilon}^*)}}\right) - \frac{t+s}{\sqrt{2}}\right)\right)\Bigg|_{t=0,s=0}. \tag{13}$$

5. Conclusion

We have calculated quantum probability measures (3) associated with coherent (8) and excited (9) states of the driven oscillator with time-dependent frequency (12), (13). The propagator for time evolution of quantum probability measures is obtained in the explicit form (11). The connection of quantum probability measures with quantum tomograms is given by the formula (5).

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