# Positive distribution description for spin states 

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#### Abstract

We investigate a possibility of describing spin states in terms of a positive distribution function depending on continuous variables like Euler's angles. A spin state reconstruction procedure similar to the symplectic tomography is considered. A quantum evolution equation for the classical-like positive distribution function is found. Generalization to arbitrary values of angular momentum is discussed. © Published by Elsevier Science B.V.


## 1. Introduction

Initially, the main objects of quantum mechanics were complex wave functions [1] (pure states) or density matrices [2] (mixed states). Some time later, it was realized that quantum systems can be described in terms of different real quasidistribution functions, both for continuous (like coordinate and momentum) [3] and discrete (like spin) [4] observables. Recently, using the ideas of symplectic tomography [5], the evolution of a quantum system with continuous observables (namely, quadrature components of a field mode) was described [6] in terms of a classical-like equation for a positive marginal distribution function of measurable squeezed and rotated quadratures. In the special case of a rotated quadrature, this function is reduced to the marginal distribution of the homodyne output variable used in an optical tomography scheme, proposed in Ref. [7] (see also Ref. [8]) and experimentally verified in Ref. [9]. Symplectic tomography and a classical distribution for a Paul trap were discussed in Ref. [10]. The equations determining energy levels of a quantum system in terms of a classical distribution were proposed and solved for a harmonic oscillator in Ref. [11].

On the other hand, "discrete" observables like spin or an angular momentum are as important in quantum mechanics as "continuous" ones. The tomography scheme for discrete variables was considered in Ref. [12]. The aim of our article is to construct using the same principle as in Ref. [5] a positive marginal distribution for rotated spin variables and to obtain an evolution equation for this function, which could be considered as a classical-like counterpart to the quantum Liouville equation. In the following section, we consider the spin-1/2 case, and after this we discuss possible generalizations to an arbitrary angular momentum.

## 2. Rotated spin marginal distribution

A generic spin state is described by a $2 \times 2$ Hermitian trace class density matrix

$$
\rho=\left(\begin{array}{ll}
\rho_{11} & \rho_{12} \\
\rho_{21} & \rho_{22}
\end{array}\right), \quad \rho_{11}+\rho_{22}=1
$$

The first problem which we address is how to determine the coefficients $\rho_{i j}$ from experimental data. The measured quantities are the mean values of the spin projections to some directions in space, determined by the unit vectors like $n=(\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \varphi)$. The average value of the spin projection to the direction $n$ equals ( $\sigma_{n} \equiv \sigma_{\alpha} n_{\alpha}, \sigma_{\alpha}$ being Pauli's matrices, $\alpha=1,2,3$ )

$$
\begin{equation*}
f(\vartheta, \varphi) \equiv \operatorname{Tr}\left(\rho \sigma_{n}\right)=\left(\rho_{11}-\rho_{22}\right) \cos \vartheta+\sin \vartheta\left(\rho_{12} \mathrm{e}^{\mathrm{i} \varphi}+\rho_{21} \mathrm{e}^{-\mathrm{i} \varphi}\right) . \tag{1}
\end{equation*}
$$

Since we have three independent parameters ( $\rho_{11}$ and $\rho_{22}$ are real, whereas $\rho_{12}=\rho_{21}^{*}$ ), one can find them choosing three different directions in space and solving some linear set of algebraic equations with respect to $\rho_{i j}$. Various schemes in this line (including the case of $N \times N$ matrices) were considered in Ref. [13]. However, in such an approach, we meet with the problem of an optimal choice of "significant" directions.

We propose another method, in which all possible directions are considered on an equal footing. Suppose that the function $f_{+}(\vartheta, \varphi)=(1+f) / 2$ (which is nothing but the probability of finding the positive value $+1 / 2$ of the spin projection to the direction $n$ ) is known for $0 \leqslant \vartheta \leqslant \pi, 0 \leqslant \varphi \leqslant 2 \pi$. Then, following the ideas of optical or symplectic tomography, the density matrix can be determined with the aid of some integral transform of the form ( $\mathrm{d} \Omega=\sin \boldsymbol{\vartheta} \mathrm{d} \boldsymbol{\vartheta} \mathrm{d} \varphi$ )

$$
\begin{equation*}
\rho=\int \mathscr{A}(\vartheta, \varphi) f_{+}(\vartheta, \varphi) \mathrm{d} \Omega, \tag{2}
\end{equation*}
$$

where $\mathscr{A}(\vartheta, \varphi)$ is a matrix of the same dimension ( $2 \times 2$ ) as matrix $\rho$. Since the two-dimensional case is highly degenerate, there exists an infinite family of admissible $\mathscr{A}$-matrices. For example, the difference $\rho_{11}-\rho_{22}$ (and, consequently, any of the values $\rho_{11}$ or $\rho_{22}$ ) can be obtained if one multiplies both sides of Eq. (1) by an arbitrary odd function of $\cos \vartheta$ and integrates over $\vartheta$ and $\varphi$, whereas to determinc the coefficient $\rho_{12}$ one could multiply both sides of (1) by the product of $\exp (-i \varphi)$ with any function $g(\vartheta)$ satisfying the condition $\int g(\vartheta) \sin (\boldsymbol{\vartheta}) \mathrm{d} \Omega \neq 0$. We give here only one explicit example,

$$
\mathscr{A}(\vartheta, \varphi)=\left(\begin{array}{cc}
\frac{1}{8 \pi}+\frac{3 \cos \vartheta}{4 \pi} & \frac{2}{\pi^{2}} \mathrm{e}^{-\mathrm{i} \varphi}  \tag{3}\\
\frac{2}{\pi^{2}} \mathrm{e}^{\mathrm{i} \varphi} & \frac{1}{8 \pi}-\frac{3 \cos \vartheta}{4 \pi}
\end{array}\right)
$$

The spin dynamics is determined by a generic Hamiltonian

$$
H=\left(\begin{array}{cc}
a & b \\
b^{*} & c
\end{array}\right)
$$

resulting in the equations

$$
\begin{equation*}
\dot{\rho}_{11}=-\dot{\rho}_{22}=\mathrm{i}\left(\rho_{12} b^{*}-\rho_{12}^{*} b\right), \quad \dot{\rho}_{12}=\mathrm{i} b\left(\rho_{11}-\rho_{22}\right)+\mathrm{i} \rho_{12}(c-a) . \tag{4}
\end{equation*}
$$

Differentiating Eq. (1) with respect to time, replacing $\dot{\rho}_{i j}$ by the expressions given in Eq. (4), and taking into account (2) and (3), we obtain the following integral equation for the positive "classical probability function" $f_{+}(\vartheta, \varphi, t)$,

$$
\begin{align*}
\dot{f}_{+}(\vartheta, \varphi, t)= & \int \mathrm{d} \Omega^{\prime} f_{+}\left(\vartheta^{\prime}, \varphi^{\prime}, t\right)\left[\frac{2}{\pi 2}(a-c) \sin \vartheta \sin \left(\varphi-\varphi^{\prime}\right)+\frac{4}{\pi^{2}} \cos \vartheta \operatorname{Im}\left(b \mathrm{e}^{\mathrm{i} \varphi^{\prime}}\right)\right. \\
& \left.-\frac{3}{2 \pi} \sin \vartheta \cos \vartheta^{\prime} \operatorname{Im}\left(b \mathrm{e}^{\mathrm{i} \varphi}\right)\right] . \tag{5}
\end{align*}
$$

## 3. Arbitrary spin

A generalization of the above construction to higher angular momenta (or multilevel systems) is straightforward. Instead of the single function (1), the starting point will be a set of functions (an ensemble of marginal distributions in rotated frames of reference)

$$
f^{(\nu)}(\vartheta, \varphi)=\operatorname{Tr}\left(\rho \mathscr{P}_{n}^{(j)}\right),
$$

where $\mathscr{P}_{n}^{(j)}$ is a projector to the state with angular momentum projection $j$ to the direction $n$. Then Eq. (2) will be generalized as follows,

$$
\rho=\sum_{j} \int \not \mathscr{X}_{j}(\vartheta, \varphi) f_{+}^{(j)}(\vartheta, \varphi) \mathrm{d} \Omega .
$$

To find the transformation matrices $\mathscr{A}_{j}$, it could be more convenient to introduce, instead of two angles $\vartheta, \varphi$, three Euler's angles ( $\phi, \theta, \psi$ ) and to calculate the convolutions of the marginal distributions with the products of Wigner's $D$-functions (which are matrix elements of the irreducible representations of the rotation group). In this case, one could use the orthogonality property of the $D$-functions with respect to the integration over the invariant measure of the rotation group.

Let us have a general relation of the Hermitian matrix $\rho$ with the transformed matrix $\rho(u)$ by means of the unitary matrix $D(u)$, where the transformation depends on extra parameters labeled by a matrix $u$. We have

$$
\begin{equation*}
\rho(u)=D \rho D^{\dagger} . \tag{6}
\end{equation*}
$$

Then for diagonal matrix elements of the Hermitian matrix $\rho(u)$ we get

$$
\begin{equation*}
\rho_{i i}(u)=D_{i s} \rho_{s m} D_{m i}^{\dagger} \tag{7}
\end{equation*}
$$

Here we mean the summation over repeated indices $s, m$ and no summation over indices $i$.
A general problem arises. Is it possible to reconstruct the matrix elements $\rho_{s m}$ from the known matrix elements $\rho_{i i}(u)$ and the matrix elements of the transformation matrix $D(u)$ ? We show that for the particular case of the states, which are the states of an arbitrary spin $j=0,1 / 2, \ldots$, the $2 \times 2$-unitary matrix $u$ may be taken in the form

$$
u(\phi, \theta, \psi)=\left(\begin{array}{cc}
\mathrm{e}^{-\mathrm{i} \phi / 2} \cos (\theta / 2) \mathrm{e}^{-\mathrm{i} \psi / 2} & -\mathrm{e}^{-\mathrm{i} \phi / 2} \sin (\theta / 2) \mathrm{e}^{\mathrm{i} \psi / 2}  \tag{8}\\
\mathrm{e}^{\mathrm{i} \phi / 2} \sin (\theta / 2) \mathrm{e}^{-\mathrm{i} \psi / 2} & \mathrm{e}^{\mathrm{i} \phi / 2} \cos (\theta / 2) \mathrm{e}^{\mathrm{i} \psi / 2}
\end{array}\right)
$$

i.e., it is parametrized by Euler's angles, and the transformation $D(u)$ has the matrix elements coinciding with the Wigner $D_{i s}^{j}(u)$-functions [14], in which $i, s=-j,-j+1, \ldots, j$, the problem has a solution.

We rewrite (7) for the particular case of the spin $j$ density matrix $\rho_{s m}^{j}$

$$
\begin{equation*}
\rho_{i i}^{j}(u)=D_{i s}^{j} \rho_{s m}^{j} D_{i m}^{j *} . \tag{9}
\end{equation*}
$$

The structure of (9) shows that, in fact, the diagonal matrix elements (9) depend on two Euler angles. From (9), in which we have made precise the notation for the spin- $j$ case, after multiplication of the equation by the Wigner $D_{n p}^{j_{2}}$-function, in which $j_{2}, n, p$ are arbitrary indices, one gets

$$
\begin{equation*}
\rho_{i i}^{j}(u) D_{n p}^{j_{2}}=D_{i s}^{j} D_{i m}^{j *} D_{n p}^{j_{2}} \rho_{s m}^{j} . \tag{10}
\end{equation*}
$$

Then we integrate equality (10) over the rotation group measure and, in view of the equality obtained, one can get $\rho_{s m}^{j}$ in terms of diagonal elements $\rho_{i i}^{j}(u)$.

We formulate the result. Given the positive marginal distribution $w(i, \phi, \theta, \psi) \equiv \rho_{i i}(u)$, the argument of which is the rotated spin $z$-projection in the fixed frame of reference for any spin described by the following operator (one could call it "rotated spin"),

$$
\begin{equation*}
\hat{J}_{z}(\phi, \theta, \psi) \equiv D_{11}^{1}(\phi, \theta, \psi) \hat{J}_{x}+D_{10}^{1}(\phi, \theta, \psi) \hat{J}_{y}+D_{1-1}^{1}(\phi, \theta, \psi) \hat{J}_{z} \tag{11}
\end{equation*}
$$

Here the parameters $\phi, \theta, \psi$ are Euler's angles and the functions $D_{1 m}^{1}(\phi, \theta, \psi)$ are Wigner $D$-function [14]. The marginal distribution is normalized,

$$
\begin{equation*}
\sum_{i=-j}^{j} w(i, \phi, \theta, \psi)=1 \tag{12}
\end{equation*}
$$

The observable $J_{z}(\phi, \theta, \psi)$, which is eigenvalue of operator (11), may be equivalently considered as $z$-spin projection in the rotated frame of reference with the inverse rotation $u^{-1}$. Then the Hermitian density matrix elements $\rho_{m m^{\prime}}^{j}$ describing the spin state are reconstructed in terms of the measurable marginal distribution $w(i, \phi, \theta, \psi)$, in which $i=-j,-j+1, \ldots, j$, due to the relation

$$
\begin{align*}
\rho_{m m^{\prime}}^{j}= & \frac{(-1)^{-(m+2 j)}}{2 \pi^{2}} \sum_{j_{2}=0}^{2 j} \sum_{l=-j_{2}}^{j_{2}}\left(2 j_{2}+1\right)^{-1} \sum_{i=-j}^{j} \int \mathrm{~d} \tilde{\Omega}(-1)^{i} w(i, \phi, \theta, \psi) \\
& \times D_{0,-1}^{j_{2}}(\phi, \theta, \psi) C_{i=10}^{j j_{2} C_{m-m^{\prime}} C_{m}^{j j_{2}} .} \tag{13}
\end{align*}
$$

Here $m, m^{\prime}=-j,-j+1, \ldots, j$ and $C_{m_{1} m_{2} m}^{j_{1} j_{j} j}$ are Clebsch-Gordan coefficients, the Euler angles vary in the domain

$$
0 \leqslant \phi<2 \pi, \quad 0 \leqslant \theta<\pi, \quad 0 \leqslant \psi<2 \pi,
$$

and

$$
\int \mathrm{d} \tilde{\Omega}=\frac{1}{8} \int_{0}^{2 \pi} \mathrm{~d} \phi \int_{0}^{\pi} \sin \theta \mathrm{d} \theta \int_{0}^{2 \pi} \mathrm{~d} \psi
$$

We used the relation [14]

$$
\begin{equation*}
\int \mathrm{d} \tilde{\Omega} D_{m^{\prime} m}^{j_{i}}(u) D_{m_{1}^{\prime} m_{1}}^{j_{1}}(u) D_{m_{2} m_{2}}^{j_{2}}(u)=\frac{2 \pi^{2}}{2 j+1} C_{m_{1}^{2} m_{2} m}^{j j_{2} j} C_{m_{1} m_{2} m}^{j_{j} j_{2} j} \tag{14}
\end{equation*}
$$

and the orthogonality properties of the Clebsch-Gordan cocfficients. The explicit form of the $D$-function in terms of Jacobi polynomials and tables of Clebsch-Gordan coefficients are given in Ref. [14]. Thus, formula (13), which is the inverse of (9), is the final result of our analysis. Given the measurable marginal distribution for arbitrary spin (9) one reconstructs the density matrix of state (13). One could conclude that along with considering the wave function of the continuous argument in the symplectic tomography approach it is possible to describe equivalently the spin states by classical distributions of a discrete variable instead of spinors.

A possible disadvantage of the proposed approach is a complicated integral evolution equation (5). Perhaps, this is the price one ought to pay for the possibility to describe essentially quantum objects like spin in terms of classical-like positive distributions of discrete variables in addition depending on extra continuous parameters. It should be pointed out that for a particular spin state like the $1 / 2$-spin, there exist a lot of possible representations of the density matrix in the form of an integral over rotation angles. The representation (13) is unique because it has an invariant form, which does not depend on the value of the spin. Also it is worth noting that the introduced marginal distribution and representation of the density matrix of an arbitrary spin state (13) are essentially different from those studied in Ref. [12], because the physical meaning of the measurable observable is different from the observable used in Ref. [12]. We used as observable the $z$-projection of the spin measured in an ensemble of rotated frames of reference. This distinguishes our approach from the approach of Ref. [12].

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