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Gauge transformation of quantum states in probability representation

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Abstract

The gauge invariance of the evolution equations of tomographic probability distribution functions of quantum particles in an electromagnetic field is illustrated. Explicit expressions for the transformations of ordinary tomograms of states under a gauge transformation of electromagnetic field potentials are obtained. Gauge-independent optical and symplectic tomographic quasi-distributions and tomographic probability distributions of states of quantum system are introduced, and their evolution equations have the Liouville equation in corresponding representations as the classical limits are found.

Keywords: quantum tomography, gauge invariance, evolution equation, optical tomogram, symplectic tomogram

1. Introduction

Gauge invariance is a fundamental quality of classical field theory and quantum electrodynamics [1, 2], as well as of Yang–Mills theory [3]. In quantum mechanics the gauge transformation makes the specific change [4] of the wave function [5] phase.

At first gauge invariance was uncovered in classical electrodynamics. The global gauge invariance leads to the law of conservation of electric charge due to Noether's theorem. In gauge-invariant theories all observable quantities, such as energy levels and cross sections of various processes calculated using the gauge-transformed and source fields, are the same.

For the formulation of quantum mechanics in phase space many scientists suggested different kinds of quasidistributions to represent the quantum states. For example, the Wigner function [6], Blohintsev function [7], Glauber–Sudarshan $P-$function [8, 9], and Husimi $Q-$function [10] can be effectively used to formulate the quantum evolution and obtain the energy levels of quantum states. All these quasidistributions are related to wave function or density matrix by integral transformations.

The Wigner function has been proposed to describe the quantum states by analogy with the description of classical states, taking into account fluctuations of the position and momentum of non-negative density distribution in the phase space of the system. Using the Wigner function in a series of works [11–14] the problem of quantum slightly non-ideal gas of uncharged particles with an interaction was first considered assuming satisfaction of the conditions of applicability of perturbation theory, and the oscillations in a Fermi liquid were explored [15]. However, the Wigner function can be negative, and therefore, it is not a probability distribution.

In [16] the probability representation of quantum mechanics was introduced (see, e.g. [17]), in which the quantum states are described by fair probabilities, called quantum tomograms. Different kinds of tomograms, e.g. optical tomograms [18, 19], symplectic tomograms [20], and spin-tomograms [21, 22] give the realization of star-product quantization schemes based on existence of specific quantizer and dequantizer operators [23, 24]. The star-product schemes bring about the constructions for non-commutative algebra of the Wigner–Weyl symbols of operators acting on a Hilbert space (see, e.g. [25–27]).

The evolution equation and energy spectrum equation for an optical tomogram were obtained in [28]. The evolution equation for a symplectic tomogram was obtained in [16]. On the other hand, the gauge properties known for the Schrödinger equation for the wave function and Moyal equation for the Wigner function [29] have not been considered until now in the tomographic representation of quantum mechanics, while gauge invariance of the Wigner–Moyal representation has been studied in sufficient detail [30–33].

Tomographic representation of quantum mechanics is completely equivalent to other well-known representations. The main advantage of tomographic representation is a classical-like appearance. In the absence of the electro-magnetic field, when considering the evolution of particles in quantum scalar potentials the dynamical equation for tomograms in the classical limit becomes the classical Liouville equation in the tomographic representation [28, 34]. But in the presence of the electro-magnetic field the classical Liouville equation does not depend on the field potentials and their gauge. Therefore, if the tomographic representation is positioned as a classical-like description, it is desirable that the quantum dynamical equation and the function characterizing the quantum state would be independent on the gauge.

In such a context the aim of our paper is to explore the gauge properties of quantum tomograms, including the star-product aspects, to introduce the gauge-independent tomograms, and to obtain evolution equations of quantum states in gauge-independent tomographic representations.

To begin with, let us recall how the gauge invariance of non-relativistic quantum mechanics is realized in the wave function or density matrix representation, and give a reminder of the basic formulae of conversion of the wave function and density matrix of a particle under the gauge transformation of the potentials of the electro-magnetic field.

Consider the motion of a quantum particle having a spin in the electromagnetic field with the vector potential $\mathbf{A}(\mathbf{q}, t)$ and the scalar potential $\varphi(\mathbf{q}, t)$. As is known, the Hamiltonian of such a system has the form [4]

$$\hat{H} = \frac{1}{2m}\left(\hat{\mathbf{P}} - \frac{e}{c}\mathbf{A}\right)^2 + e\varphi - \hat{\boldsymbol{\kappa}}\mathbf{B}, \tag{1}$$

where $\hat{\mathbf{P}} = -\mathrm{i}\hbar\partial/\partial\mathbf{q}$ is a generalized momentum operator, m and e are mass and charge of the particle, $\mathbf{B} = \mathrm{rot}\mathbf{A}$ is a magnetic field strength, $\hat{\boldsymbol{\kappa}}$ is an operator of quantum-mechanical magnetic moment

$$\hat{\boldsymbol{\kappa}} = \frac{\kappa}{s}\hat{\mathbf{s}}, \tag{2}$$

where s is a spin of the particle, $\hat{\mathbf{s}}$ is a spin operator, and κ is a constant characteristic of the particle (the value of the intrinsic magnetic moment) that is the highest possible modulo value κ_3 of projection of the magnetic moment on the q_3 axis achieved with the projection of the spin on this axis equal to s.

From the classical electrodynamics it is known that potentials of the field are defined only up to the gauge transformation [1]

$$\mathbf{A} \to \mathbf{A} + \nabla\chi, \quad \varphi \to \varphi - \frac{1}{c}\frac{\partial\chi}{\partial t}, \tag{3}$$

where χ is an arbitrary function of spatial coordinates and time.

Since the electric field intensity $\mathbf{E}$ and the magnetic field strength $\mathbf{B}$ are defined in terms of the potentials as:

$$\mathbf{E} = -\mathrm{grad}\varphi - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}, \quad \mathbf{B} = \mathrm{rot}\mathbf{A}, \tag{4}$$

then the gauge transformation (3) does not affect the values of $\mathbf{E}$ and $\mathbf{B}$. Therefore the part of Hamiltonian (1) responsible for the interaction of the spin with the magnetic field is independent of the gauge transformation.

The requirement of invariance of the Schrödinger equation under the gauge transformation simultaneously with the gauge-independence of 'probability density' $|\Psi|^2$ leads us to the form of the conversion of the wave function [4]:

$$\Psi \to \exp\left(\frac{\mathrm{i}e}{c\hbar}\chi\right)\Psi. \tag{5}$$

Accordingly, the conversions of the density matrix of the state and the Hamiltonian of the system under the gauge transformation acquire the forms:

$$\hat{\rho}_{\mathrm{c}} = \exp\left(\frac{\mathrm{i}e}{c\hbar}\chi\right)\hat{\rho}\,\exp\left(-\frac{\mathrm{i}e}{c\hbar}\chi\right), \tag{6}$$

$$\hat{H}_{\mathrm{c}} = \exp\left(\frac{\mathrm{i}e}{c\hbar}\chi\right)\hat{H}\,\exp\left(-\frac{\mathrm{i}e}{c\hbar}\chi\right), \tag{7}$$

and the von-Neumann equation is also invariant under transformations (3) and (6)

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\hat{\rho} = [\hat{H}, \hat{\rho}] \;\to\; \mathrm{i}\hbar\frac{\partial}{\partial t}\hat{\rho}_{\mathrm{c}} = [\hat{H}_{\mathrm{c}}, \hat{\rho}_{\mathrm{c}}]\,. \tag{8}$$

The paper is organized as follows. In section 2 we find transformations of ordinary quantum tomograms in the general case in terms of the gauge-independent quantizer and dequantizer operators. In section 3 we obtain the evolution equations for classical and quantum particles in the classical electro-magnetic field in tomographic representations with gauge-independent dequantizers, we discuss the gauge invariance of these equations and illustrate that the quantum tomographic equations do not be transformed to the corresponding classical equations when $\hbar \to 0$. In section 4 we introduce gauge-independent optical and symplectic quantum tomographic quasidistributions and derive evolution equations for such representations. In section 5 we introduce and study the gauge-independent tomographic probability representation and find the evolution equation for it. The conclusion is presented in section 6.

2. Gauge transformations of ordinary quantum tomograms

In the probability representation of quantum mechanics the states of the system are described by probability distribution functions $w(z,\eta,t)$ called quantum tomograms, where z is a set of distribution variables, η is a set of parameters of corresponding tomography, and t is time. According to the universal star-product scheme (see [26]), the tomograms are introduced as the average values of dequantizer operators $\hat{U}(z,\eta)$,

$$w(z,\eta,t) = \mathrm{Tr}\{\hat{\rho}(t)\hat{U}(z,\eta)\}, \tag{9}$$

The inverse transformation is determined by the quantizer operator $\hat{D}(z,\eta)$

$$\hat{\rho}(t) = \int \hat{D}(z,\eta)w(z,\eta,t)\mathrm{d}z\mathrm{d}\eta. \tag{10}$$

The von-Neumann equation in the tomographic representation has the form [34]:

$$\frac{\partial}{\partial t}w(z,\eta,t) = \frac{2}{\hbar}\int \mathrm{Im}\,[\mathrm{Tr}\{\hat{H}(t)\hat{D}(z',\eta')\hat{U}(z,\eta)\}]\,w(z',\eta',t)\mathrm{d}z'\mathrm{d}\eta'. \tag{11}$$

It is easy to see that if we determine in definition (9) that the dequantizer and the quantizer are gauge-independent, then equation (11) is invariant under the gauge transformation only with the following transformation of tomograms:

$$\begin{aligned} w(z,\eta,t) \to w_{\mathrm{c}}(z,\eta,t) &= \mathrm{Tr}\left\{\exp\left(\frac{\mathrm{i}e}{c\hbar}\chi\right)\hat{\rho}(t)\exp\left(-\frac{\mathrm{i}e}{c\hbar}\chi\right)\hat{U}(z,\eta)\right\} \\ &= \mathrm{Tr}\left\{\exp\left(\frac{\mathrm{i}e}{c\hbar}\chi\right)\int \hat{D}(z',\eta')w(z',\eta',t)\mathrm{d}z'\mathrm{d}\eta'\exp\left(-\frac{\mathrm{i}e}{c\hbar}\chi\right)\hat{U}(z,\eta)\right\}. \end{aligned} \tag{12}$$

Introducing the notation for the kernel $G(z,\eta,z',\eta')$

$$G(z,\eta,z',\eta') = \mathrm{Tr}\left\{\exp\left(\frac{\mathrm{i}e}{c\hbar}\chi\right)\hat{D}(z',\eta')\exp\left(-\frac{\mathrm{i}e}{c\hbar}\chi\right)\hat{U}(z,\eta)\right\}, \tag{13}$$

for the gauge transformed function $w_{\mathrm{c}}(z,\eta,t)$ we obtain

$$w_{\mathrm{c}}(z,\eta,t) = \int G(z,\eta,z',\eta')w(z',\eta',t)\mathrm{d}z'\mathrm{d}\eta'. \tag{14}$$

Thus, under the gauge transformation of the electromagnetic field potentials the tomogram of the state is converted by means of integral transformation (14), in which the explicit form of the kernel depends on the type of tomography.

If we have a spinless quantum particle with mass m in three-dimensional space, then the dequantizer and the quantizer for the optical tomography have the form [35]

$$\hat{U}_w(\mathbf{X},\boldsymbol{\theta}) = \prod_{\sigma=1}^{3}\delta\left(X_\sigma - \hat{q}_\sigma\cos\theta_\sigma - \hat{P}_\sigma\frac{\sin\theta_\sigma}{m\omega_\sigma}\right), \tag{15}$$

$$\hat{D}_w(\mathbf{X},\boldsymbol{\theta}) = \int\prod_{\sigma=1}^{3}\frac{\hbar|k_\sigma|}{2\pi m\omega_\sigma}\exp\left\{\mathrm{i}k_\sigma\left(X_\sigma - \hat{q}_\sigma\cos\theta_\sigma - \hat{P}_\sigma\frac{\sin\theta_\sigma}{m\omega_\sigma}\right)\right\}\mathrm{d}^3k, \tag{16}$$

where $\hat{P}_\sigma$ are components of the generalized momentum operator and ω_σ are constants that have the dimension of frequency. Further, for simplicity we choose the set $\{\omega_\sigma\}$ so that $\omega_1 = \omega_2 = \omega_3 = \omega$.

Substituting these expressions of the dequantizer and the quantizer to equation (13), after some calculations using the formula for the matrix elements

$$\langle q'_\sigma|e^{\mathrm{i}(a\hat{q}_\sigma+b\hat{P}_\sigma)}|q_\sigma\rangle = \mathrm{e}^{\mathrm{i}a(q_\sigma+q'_\sigma)/2}\delta(q'_\sigma - q_\sigma + b), \tag{17}$$

we obtain

$$\begin{aligned} &G_w(\mathbf{X},\boldsymbol{\theta},\mathbf{X}',\boldsymbol{\theta}') \\ &= \frac{1}{(4\pi^2\hbar)^3}\int \exp\left\{\frac{\mathrm{i}e}{c\hbar}\left[\chi\left(\frac{k_\sigma\sin\theta_\sigma}{m\omega}+\frac{\sqrt{\hbar}\,\sin\theta'_\sigma}{2\sqrt{m\omega}}r_\sigma\right)-\chi\left(\frac{k_\sigma\sin\theta_\sigma}{m\omega}-\frac{\sqrt{\hbar}\,\sin\theta'_\sigma}{2\sqrt{m\omega}}r_\sigma\right)\right]\right\} \\ &\quad\times\prod_{\sigma=1}^{3}|r_\sigma|\exp\left\{\mathrm{i}r_\sigma\sqrt{\frac{m\omega}{\hbar}}\left(X'_\sigma - X_\sigma\frac{\sin\theta'_\sigma}{\sin\theta_\sigma}\right)-\mathrm{i}k_\sigma r_\sigma\frac{\sin(\theta_\sigma-\theta'_\sigma)}{\sqrt{m\omega\hbar}}\right\}\mathrm{d}^3k\,\mathrm{d}^3r. \end{aligned} \tag{18}$$

Further simplification of this expression is unfortunately only possible with the explicit expression for the function χ.

For the spinless symplectic tomography the dequantizer and quantizer are given by the formulae

$$\hat{U}_M(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu}) = \prod_{\sigma=1}^{3}\delta(X_\sigma - \hat{q}_\sigma\mu_\sigma - \hat{P}_\sigma\nu_\sigma), \tag{19}$$

$$\hat{D}_M(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu}) = \prod_{\sigma=1}^{3}\frac{m\omega}{2\pi}\exp\left\{\mathrm{i}\sqrt{\frac{m\omega}{\hbar}}(X_\sigma - \hat{q}_\sigma\mu_\sigma - \hat{P}_\sigma\nu_\sigma)\right\}, \tag{20}$$

and from (13) we can obtain

$$\begin{aligned} &G_M(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu},\mathbf{X}',\boldsymbol{\mu}',\boldsymbol{\nu}') \\ &= \left(\frac{m\omega}{4\pi^2\hbar}\right)^3\int\exp\left\{\frac{\mathrm{i}e}{c\hbar}\left[\chi\left(\nu_\sigma k_\sigma+\frac{\sqrt{m\omega\hbar}}{2}\nu'_\sigma\right)-\chi\left(\nu_\sigma k_\sigma-\frac{\sqrt{m\omega\hbar}}{2}\nu'_\sigma\right)\right]\right\} \\ &\quad\times\prod_{\sigma=1}^{3}\exp\left\{\mathrm{i}\sqrt{\frac{m\omega}{\hbar}}\left[k_\sigma(\mu\nu'_\sigma-\mu'_\sigma\nu)+X'_\sigma - X_\sigma\frac{\nu'_\sigma}{\nu_\sigma}\right]\right\}\mathrm{d}^3k. \end{aligned} \tag{21}$$

Note that the kernels G_w and G_M are connected by the relation

$$G_w(\mathbf{X},\boldsymbol{\theta},\mathbf{X}',\boldsymbol{\theta}') = \int\frac{|r_1|\,|r_2|\,|r_3|}{(m\omega)^3}G_M\left(X_\sigma,\cos\theta_\sigma,\frac{\sin\theta_\sigma}{m\omega},r_\sigma X',r_\sigma\cos\theta'_\sigma,r_\sigma\frac{\sin\theta'_\sigma}{m\omega}\right)\mathrm{d}^3r.$$

Consider now the positive vector non-redundant tomography of the particle with spin [36, 37]

$$\mathbf{w}(z,\eta,t) = \mathrm{Tr}\{\hat{\rho}(t)\hat{\mathbf{U}}(z,\eta)\}, \tag{22}$$

where the trace is calculated also over spin indexes. In this representation the components of the dequantizer and the quantizer are defined by formulae

$$\hat{U}_{j(nl)}(z,\eta) = \hat{U}(z,\eta)\otimes\hat{\mathcal{U}}_{j(nl)},\quad \hat{D}_{(nl)j}(z,\eta) = \hat{D}(z,\eta)\otimes\hat{\mathcal{D}}_{(nl)j}, \tag{23}$$

where $\hat{\mathcal{U}}_{j(nl)}$ and $\hat{\mathcal{D}}_{(nl)j}$ are the spin dequantizer and quantizer, $j=\overline{1,\,(2s+1)^2}$ is the index corresponding to the jth component of the vector tomogram $\mathbf{w}(z,\eta,t)$, and $n,l=\overline{1,\,(2s+1)}$ are the spin indexes. Since

$$\sum_{n,l=1}^{2s+1} \hat{\mathcal{D}}_{(nl)j'} \hat{\mathcal{U}}_{j(ln)} = \delta_{jj'}, \tag{24}$$

then, according to general formula (13), the kernel of the transformation of the vector tomogram $\mathbf{w}(z, \eta, t)$ takes the form:

$$\begin{aligned} G_{jj'}(z, \eta, z', \eta') &= \mathrm{Tr}\left\{\exp\left(\frac{\mathrm{i}e}{c\hbar}\chi\right)\hat{D}(z', \eta') \otimes \hat{\mathcal{D}}_{j'} \exp\left(-\frac{\mathrm{i}e}{c\hbar}\chi\right)\hat{U}(z, \eta) \otimes \hat{\mathcal{U}}_j\right\} \\ &= \delta_{jj'} G(z, \eta, z', \eta'), \end{aligned} \tag{25}$$

i.e. the vector tomogram $\mathbf{w}(z, \eta, t)$ under the gauge transformation is converted by components through the integral transformation:

$$\mathbf{w}(z, \eta, t) \;\rightarrow\; \mathbf{w}_{\mathrm{c}}(z, \eta, t) = \int G(z, \eta, z', \eta')\mathbf{w}(z', \eta', t)\mathrm{d}z'\mathrm{d}\eta', \tag{26}$$

where $G(z, \eta, z', \eta')$ is a kernel of the integral transformation for the spinless case. This formula is valid for an arbitrary spin.

Thus we see that if the dequantizer is gauge-independent, then the tomogram is gauge-dependent, and the evolution equation is gauge-invariant but gauge-dependent.

3. Gauge invariance of evolution equations

Let us consider in more detail the gauge invariance of the quantum evolution equations in the tomographic representations with gauge-independent dequantizers and the question of limiting transition of such equations to classics when $\hbar \rightarrow 0$. At first, we will get the Liouville equation in the electro-magnetic field in the tomographic representations.

For the classical ensemble of non-interacting particles with mass m and charge e this equation in the phase space has the form:

$$\frac{\partial}{\partial t} W_{\mathrm{cl}}(\mathbf{q}, \mathbf{p}, t) + \frac{\mathbf{p}}{m}\frac{\partial}{\partial \mathbf{q}} W_{\mathrm{cl}}(\mathbf{q}, \mathbf{p}, t) + e\left(\mathbf{E}(\mathbf{q}, t) + \frac{1}{mc}[\mathbf{p} \times \mathbf{B}(\mathbf{q}, t)]\right)\frac{\partial}{\partial \mathbf{p}} W_{\mathrm{cl}}(\mathbf{q}, \mathbf{p}, t) = 0, \tag{27}$$

where $\mathbf{p}$ is a kinetic momentum, $\mathbf{E}(\mathbf{q}, t)$ and $\mathbf{B}(\mathbf{q}, t)$ are electric and magnetic fields, defined by formulae (4), $W_{\mathrm{cl}}(\mathbf{q}, \mathbf{p}, t)$ is a distribution function of non-interacting particles.

The distribution function $W_{\mathrm{cl}}(\mathbf{q}, \mathbf{p}, t)$ is independent on the gauge transformation [1], because the Liouville equation (27) includes only gauge-independent intensities of the electromagnetic field. Consequently, the optical and symplectic tomograms of the function $W_{\mathrm{cl}}(\mathbf{q}, \mathbf{p}, t)$ defined by the formulae (see [28])

$$w_{\mathrm{cl}}(\mathbf{x}, \boldsymbol{\theta}, t) = \int W_{\mathrm{cl}}(\mathbf{q}, \mathbf{p}, t) \prod_{\sigma=1}^{3} \delta\left(x_\sigma - q_\sigma \cos\theta_\sigma - p_\sigma \frac{\sin\theta_\sigma}{m\omega}\right)\mathrm{d}^3q\, \mathrm{d}^3p, \tag{28}$$

$$M_{\mathrm{cl}}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu}, t) = \int W_{\mathrm{cl}}(\mathbf{q}, \mathbf{p}, t) \prod_{\sigma=1}^{3} \delta(x_\sigma - \mu_\sigma q_\sigma - \nu_\sigma p_\sigma)\mathrm{d}^3q\, \mathrm{d}^3p, \tag{29}$$

are also independent on the gauge transformation. We use the designation $\mathbf{x}$ instead of $\mathbf{X}$ for the distribution variable to point out that the Radon transformations (28) and (29) are made in the phase space with kinetic momentum $\mathbf{p}$.

Using the known correspondence rules [28, 35] between the operators acting on the Wigner function [6] (or on the distribution function) and the operators acting on the optical or symplectic tomograms

$$\begin{aligned}
q_\sigma W(\mathbf{q},\mathbf{p}) &\leftrightarrow -\partial_{x_\sigma}^{-1}\partial_{\mu_\sigma}M(\mathbf{x},\boldsymbol{\mu},\boldsymbol{\nu}) &&\leftrightarrow \Big(\sin\theta_\sigma\partial_{x_\sigma}^{-1}\partial_{\theta_\sigma}+x_\sigma\cos\theta_\sigma\Big)w(\mathbf{x},\boldsymbol{\theta}),\\
p_\sigma W(\mathbf{q},\mathbf{p}) &\leftrightarrow -\partial_{x_\sigma}^{-1}\partial_{\nu_\sigma}M(\mathbf{x},\boldsymbol{\mu},\boldsymbol{\nu}) &&\leftrightarrow m\omega\Big(-\cos\theta_\sigma\partial_{x_\sigma}^{-1}\partial_{\theta_\sigma}+x_\sigma\sin\theta_\sigma\Big)w(\mathbf{x},\boldsymbol{\theta}),\\
\partial_{q_\sigma}W(\mathbf{q},\mathbf{p}) &\leftrightarrow \mu_\sigma\partial_{x_\sigma}M(\mathbf{x},\boldsymbol{\mu},\boldsymbol{\nu}) &&\leftrightarrow \cos\theta_\sigma\partial_{x_\sigma}w(\mathbf{x},\boldsymbol{\theta}),\\
\partial_{p_\sigma}W(\mathbf{q},\mathbf{p}) &\leftrightarrow \nu_\sigma\partial_{x_\sigma}M(\mathbf{x},\boldsymbol{\mu},\boldsymbol{\nu}) &&\leftrightarrow \frac{\sin\theta_\sigma}{m\omega}\partial_{x_\sigma}w(\mathbf{x},\boldsymbol{\theta}),
\end{aligned} \tag{30}$$

where we introduced the designation [35] for inverse derivatives

$$\partial_{x_\sigma}^{-n}f(x_\sigma)=\frac{1}{(n-1)\,!}\int(x_\sigma-x'_\sigma)^{n-1}\Theta(x_\sigma-x'_\sigma)f(x'_\sigma)\mathrm{d}x'_\sigma, \tag{31}$$

where $\Theta(x_\sigma-x'_\sigma)$ is a Heaviside step function, we can write the Liouville equation (27) in the optical and the symplectic tomography representation

$$\begin{aligned}
\partial_t w_{\mathrm{cl}}(\mathbf{x},\boldsymbol{\theta},t)=&\Bigg[\omega\sum_{j=1}^{3}\Bigg(\cos^2\theta_j\partial_{\theta_j}-\frac{\sin 2\theta_j}{2}\Big\{1+x_j\partial_{x_j}\Big\}\Bigg)\\
&+\frac{e}{mc}\sum_{\alpha,\beta,\gamma=1}^{3}\varepsilon_{\alpha\beta\gamma}B_\gamma(\sin\theta_\sigma\partial_{x_\sigma}^{-1}\partial_{\theta_\sigma}+x_\sigma\cos\theta_\sigma,t)\Big(\cos\theta_\beta\partial_{x_\beta}^{-1}\partial_{\theta_\beta}-x_\beta\sin\theta_\beta\Big)\sin\theta_\alpha\partial_{x_\alpha}\\
&-\frac{e}{m\omega}\sum_{j=1}^{3}E_j(\sin\theta_\sigma\partial_{x_\sigma}^{-1}\partial_{\theta_\sigma}+x_\sigma\cos\theta_\sigma,t)\sin\theta_j\partial_{x_j}\Bigg]w_{\mathrm{cl}}(\mathbf{x},\boldsymbol{\theta},t),
\end{aligned} \tag{32}$$

$$\begin{aligned}
\partial_t M_{\mathrm{cl}}(\mathbf{x},\boldsymbol{\mu},\boldsymbol{\nu},t)=&\Bigg[\frac{\boldsymbol{\mu}}{m}\partial_{\boldsymbol{\nu}}+\frac{e}{mc}\sum_{\alpha,\beta,\gamma=1}^{3}\varepsilon_{\alpha\beta\gamma}B_\gamma(-\partial_{x_\sigma}^{-1}\partial_{\mu_\sigma},t)\Big(\partial_{x_\beta}^{-1}\partial_{\nu_\beta}\Big)\nu_\alpha\partial_{x_\alpha}\\
&-e\sum_{j=1}^{3}E_j(-\partial_{x_\sigma}^{-1}\partial_{\mu_\sigma},t)\nu_j\partial_{x_j}\Bigg]M_{\mathrm{cl}}(\mathbf{x},\boldsymbol{\mu},\boldsymbol{\nu},t),
\end{aligned} \tag{33}$$

where $\varepsilon_{\alpha\beta\gamma}$ is the completely antisymmetric pseudo-tensor of 3rd rank (the Levi-Civita symbol).

Thus, we have gauge-independent equations (32) and (33) for gauge-independent classical tomograms $w_{\mathrm{cl}}(\mathbf{x},\boldsymbol{\theta},t)$ and $M_{\mathrm{cl}}(\mathbf{x},\boldsymbol{\mu},\boldsymbol{\nu},t)$.

As is known, if we have the ensemble of non-interacting particles in the potential field, the generalized momentum of the particle is equal to its kinetic momentum, and the quantum analogue of the Liouville equation in this case is the Moyal equation [38] for the Wigner function [6]

$$W(\mathbf{q},\mathbf{P},t)=\frac{1}{(2\pi\hbar)^3}\int\rho\Big(\mathbf{q}-\frac{\mathbf{u}}{2},\mathbf{q}+\frac{\mathbf{u}}{2},t\Big)\exp\Big(\frac{\mathrm{i}}{\hbar}\mathbf{uP}\Big)\mathrm{d}^3u, \tag{34}$$

which is converted into the Liouville equation when $\hbar\to 0$.

In the electro-magnetic field when $\mathbf{A}\neq 0$, the Moyal equation for the function (34) is written as follows:

$$\frac{\partial}{\partial t}W(\mathbf{q},\mathbf{P},t)=\Bigg[-\frac{\mathbf{P}}{m}\frac{\partial}{\partial\mathbf{q}}+\frac{2e}{\hbar}\operatorname{Im}\varphi\Big(\mathbf{q}+\frac{\mathrm{i}\hbar}{2}\frac{\partial}{\partial\mathbf{P}},t\Big)+\frac{e^2}{mc^2\hbar}\operatorname{Im}\mathbf{A}^2\Big(\mathbf{q}+\frac{\mathrm{i}\hbar}{2}\frac{\partial}{\partial\mathbf{P}},t\Big)$$
$$-\frac{2e}{mc\hbar}\operatorname{Im}\Big\{\mathbf{A}\Big(\mathbf{q}+\frac{\mathrm{i}\hbar}{2}\frac{\partial}{\partial\mathbf{P}},t\Big)\Big(\mathbf{P}-\frac{\mathrm{i}\hbar}{2}\frac{\partial}{\partial\mathbf{q}}\Big)\Big\}$$
$$+\frac{e}{mc}\operatorname{Re}\nabla_{\mathbf{q}}\mathbf{A}\Big(\mathbf{q}\to\mathbf{q}+\frac{\mathrm{i}\hbar}{2}\frac{\partial}{\partial\mathbf{P}},t\Big)\Bigg]W(\mathbf{q},\mathbf{P},t). \tag{35}$$

The function $W(\mathbf{q},\mathbf{P},t)$ is gauge-dependent, but if we take the classical limit $\hbar\to 0$ and change variables $\mathbf{p}=\mathbf{P}-\frac{e}{c}\mathbf{A}$ in (35), then this equation will be converted into a gauge-independent Liouville equation (27). However, there is no contradiction here, because in the gauge transformation of the function $W(\mathbf{q},\mathbf{p}+\frac{e}{c}\mathbf{A})$

$$W_{\mathrm{c}}\Big(\mathbf{q},\mathbf{p}+\frac{e}{c}\mathbf{A}\Big)=\int W\Big(\mathbf{q},\mathbf{p}'+\frac{e}{c}\mathbf{A}\Big)\exp\Big\{\frac{\mathrm{i}}{\hbar}\mathbf{u}(\mathbf{p}-\mathbf{p}')\Big\}$$
$$\times\exp\Big\{\frac{\mathrm{i}e}{c\hbar}\Big[\chi\Big(\mathbf{q}-\frac{\mathbf{u}}{2}\Big)-\chi\Big(\mathbf{q}+\frac{\mathbf{u}}{2}\Big)+\mathbf{u}\nabla\chi(\mathbf{q})\Big]\Big\}\frac{\mathrm{d}^3u\,\mathrm{d}^3p'}{(2\pi\hbar)^3}$$

we can spread out the function $\chi(\mathbf{q}\pm\mathbf{u}/2)$ up to the first order $\chi(\mathbf{q}\pm\mathbf{u}/2)\approx\chi(\mathbf{q})\pm\frac{1}{2}\mathbf{u}\nabla\chi(\mathbf{q})$ using a method of a stationary phase at $\hbar\to 0$. After that in the limit case we obtain

$$W_{\mathrm{c}}\Big(\mathbf{q},\mathbf{p}+\frac{e}{c}\mathbf{A}\Big)=\int W\Big(\mathbf{q},\mathbf{p}'+\frac{e}{c}\mathbf{A}\Big)\delta(\mathbf{p}-\mathbf{p}')\mathrm{d}^3p'=W\Big(\mathbf{q},\mathbf{p}+\frac{e}{c}\mathbf{A}\Big),$$

that is the function $W\Big(\mathbf{q},\mathbf{p}+\frac{e}{c}\mathbf{A}\Big)$ becomes gauge-independent.

Let us transform the Moyal equation (35) to the optical and symplectic tomographic representations, in which the tomograms $w(\mathbf{X},\boldsymbol{\theta},t)$ and $M(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu},t)$ are defined from the Wigner function $W(\mathbf{q},\mathbf{P},t)$ with the same formulae (28) and (29), where the kinetic momentum $\mathbf{p}$ should be replaced by the generalized momentum $\mathbf{P}$, and the variable $\mathbf{x}$ should be replaced by $\mathbf{X}$ to point out that the Radon transformations are being done in the phase space with generalized momentum. For this purpose we should use the same correspondence rules as (30). After calculations we can write the evolution equation for gauge-dependent optical tomogram as follows:

$$\partial_t w\,(\mathbf{X},\boldsymbol{\theta},t)=\Bigg[\omega\sum_{j=1}^{3}\Big(\cos^2\theta_j\partial_{\theta_j}-\frac{1}{2}\sin 2\theta_j\Big\{1+X_j\partial_{X_j}\Big\}\Big)+\frac{2e}{\hbar}\operatorname{Im}[\hat{\varphi}]_w$$
$$+\frac{e^2}{mc^2\hbar}\operatorname{Im}[\hat{\mathbf{A}}]_w^2-\frac{2e}{mc\hbar}\operatorname{Im}([\hat{\mathbf{A}}]_w[\hat{\mathbf{P}}]_w)+\frac{e}{mc}\operatorname{Re}\big[\nabla_{\mathbf{q}}\mathbf{A}\big]_w\Bigg]w(\mathbf{X},\boldsymbol{\theta},t), \tag{36}$$

where

$$[\hat{A}_j]_w=A_j([\hat{\mathbf{q}}]_w,t),\quad[\hat{\varphi}]_w=\varphi([\hat{\mathbf{q}}]_w,t),\quad[\nabla_{\mathbf{q}}\hat{\mathbf{A}}]_w=\nabla_{\mathbf{q}}\mathbf{A}(\mathbf{q}\to[\hat{\mathbf{q}}]_w,t),$$

and $[\hat{\mathbf{q}}]_w$, $[\hat{\mathbf{P}}]_w$ are position and generalized momentum operators in the optical tomographic representation [35],

$$[\hat{q}_\sigma]_w=\sin\theta_\sigma\partial_{\theta_\sigma}\partial_{X_\sigma}^{-1}+X_\sigma\cos\theta_\sigma+\mathrm{i}\frac{\hbar\sin\theta_\sigma}{2m\omega}\partial_{X_\sigma}, \tag{37}$$

$$[\hat{P}_\sigma]_w = m\omega\Big(-\cos\theta_\sigma\partial_{X_\sigma}^{-1}\partial_{\theta_\sigma} + X_\sigma\sin\theta_\sigma\Big) - \frac{\mathrm{i}\hbar}{2}\cos\theta_\sigma\partial_{X_\sigma}. \tag{38}$$

For the gauge-dependent symplectic tomogram we can write

$$\partial_t M(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu},t) = \Bigg[\frac{\boldsymbol{\mu}}{m}\partial_{\boldsymbol{\nu}} + \frac{2e}{\hbar}\,\mathrm{Im}\,[\hat{\varphi}]_M + \frac{e^2}{mc^2\hbar}\,\mathrm{Im}[\hat{\mathbf{A}}]_M^2 - \frac{2e}{mc\hbar}\,\mathrm{Im}([\hat{\mathbf{A}}]_M[\hat{\mathbf{P}}]_M) + \frac{e}{mc}\,\mathrm{Re}\Big[\nabla_{\mathbf{q}}\mathbf{A}\Big]_M\Bigg]M(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu},t), \tag{39}$$

where

$$[\hat{A}_j]_M = A_j([\hat{\mathbf{q}}]_M,t),\quad [\hat{\varphi}]_M = \varphi([\hat{\mathbf{q}}]_M,t),\quad [\nabla_{\mathbf{q}}\mathbf{A}]_M = \nabla_{\mathbf{q}}\mathbf{A}(\mathbf{q}\to[\hat{\mathbf{q}}]_M,t),$$

and $[\hat{\mathbf{q}}]_M, [\hat{\mathbf{P}}]_M$ are position and generalized momentum operators in the symplectic representation (see [28]),

$$[\hat{P}_\sigma]_M = -\partial_{X_\sigma}^{-1}\partial_{\nu_\sigma} - \mathrm{i}(\hbar/2)\mu_\sigma\partial_{X_\sigma},\quad [\hat{q}_\sigma]_M = -\partial_{X_\sigma}^{-1}\partial_{\mu_\sigma} + \mathrm{i}(\hbar/2)\nu_\sigma\partial_{X_\sigma}. \tag{40}$$

Equations (36) and (39) are gauge-invariant only under the condition of transformation of tomograms with general formula (14) with the kernel $G(z,\eta,z',\eta')$ defined by the formula (18) or (21). In the classical limit $\hbar\to 0$ these equations, in the general case, are not converted into equations (32) and (33). The thing is that (32) and (36) are equations for distribution functions of different observables: $x_\sigma(\theta_\sigma) = q_\sigma\cos\theta_\sigma + p_\sigma\sin\theta_\sigma$ in the classical case (32); but $\hat{X}_\sigma(\theta_\sigma) = \hat{q}_\sigma\cos\theta_\sigma + \hat{P}_\sigma\sin\theta_\sigma$ in the quantum case (36). Analogously, (33) and (39) are equations for distribution functions of different observables $x_\sigma(\mu_\sigma,\nu_\sigma) = q_\sigma\mu_\sigma + p_\sigma\nu_\sigma$ and $\hat{X}_\sigma(\mu_\sigma,\nu_\sigma) = \hat{q}_\sigma\mu_\sigma + \hat{P}_\sigma\nu_\sigma$ respectively.

4. Gauge-independent tomographic quasiprobability representations

In the previous section we have shown that the evolution equations in the tomographic representations for the gauge-dependent tomograms in the classical limit $\hbar\to 0$ are not converted to the Liouville equation in the tomographic forms for gauge-independent tomograms of the classical distribution function.

Therefore, for the construction of quantum tomographic representations, in which the evolution equations would have been transformed to (32) and (33) when $\hbar\to 0$, we need to introduce gauge-independent quantum tomograms. This can be done with the help of the gauge-independent Wigner function obtained in [29],

$$W_{\mathrm{g}}(\mathbf{q},\mathbf{p},t) = \frac{1}{(2\pi\hbar)^3}\int\exp\left(\frac{\mathrm{i}}{\hbar}\mathbf{u}\left\{\mathbf{p} + \frac{e}{c}\int_{-1/2}^{1/2}\mathrm{d}\tau\mathbf{A}(\mathbf{q}+\tau\mathbf{u},t)\right\}\right)\rho\left(\mathbf{q}-\frac{\mathbf{u}}{2},\mathbf{q}+\frac{\mathbf{u}}{2},t\right)\mathrm{d}^3u, \tag{41}$$

where $\mathbf{p}$ is a kinetic momentum.

The gauge-independent Moyal equation for this function has the form [30]:

$$\left\{\partial_t + \frac{1}{m}(\mathbf{p}+\triangle\tilde{\mathbf{p}})\partial_{\mathbf{q}} + e\left(\tilde{\mathbf{E}} + \frac{1}{mc}[(\mathbf{p}+\triangle\tilde{\mathbf{p}})\times\tilde{\mathbf{B}}]\right)\partial_{\mathbf{p}}\right\}W_{\mathrm{g}}(\mathbf{q},\mathbf{p},t) = 0, \tag{42}$$

where

$$\triangle\tilde{\mathbf{p}} = -\frac{e}{c}\frac{\hbar}{\mathrm{i}}\left[\frac{\partial}{\partial\mathbf{p}} \times \int_{-1/2}^{1/2} \mathrm{d}\tau\, \tau\mathbf{B}\left(\mathbf{q} + \mathrm{i}\hbar\tau\frac{\partial}{\partial\mathbf{p}},\, t\right)\right],$$

$$\tilde{\mathbf{E}} = \int_{-1/2}^{1/2} \mathrm{d}\tau\, \mathbf{E}\left(\mathbf{q} + \mathrm{i}\hbar\tau\frac{\partial}{\partial\mathbf{p}},\, t\right),\quad \tilde{\mathbf{B}} = \int_{-1/2}^{1/2} \mathrm{d}\tau\, \mathbf{B}\left(\mathbf{q} + \mathrm{i}\hbar\tau\frac{\partial}{\partial\mathbf{p}},\, t\right).$$

This equation in the classical limit $\hbar \to 0$ is converted into the Liouville equation (27).

If we apply Radon transformations (28) and (29) to the Wigner function (41), we obtain gauge-independent optical $w_{\mathrm{g}}(\mathbf{x}, \boldsymbol{\theta}, t)$ and symplectic $M_{\mathrm{g}}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu}, t)$ tomograms. Under such definitions the correspondence rules between operators acting on the tomograms and the Wigner function will be similar to the correspondence rules (30). Then, from equation (42) we find the evolution equation for the gauge-independent optical tomogram $w_{\mathrm{g}}(\mathbf{x}, \boldsymbol{\theta}, t)$:

$$\begin{aligned}\partial_t w_{\mathrm{g}}(\mathbf{x}, \boldsymbol{\theta}, t) = \Bigg[&\omega\sum_{j=1}^{3}\Bigg(\cos^2\theta_j\partial_{\theta_j} - \frac{1}{2}\sin 2\theta_j\{1 + x_j\partial_{x_j}\}\\ &- \frac{1}{m}\sum_{\alpha=1}^{3}\left[\triangle\tilde{\mathbf{p}}_\alpha\right]_w \cos\theta_\alpha\partial_{x_\alpha} - \frac{e}{m\omega}\sum_{j=1}^{3}\left[\tilde{\mathbf{E}}_j\right]_w \sin\theta_j\partial_{x_j}\Bigg)\\ &+ \frac{e}{mc}\sum_{\alpha,\beta,\gamma=1}^{3}\varepsilon_{\alpha\beta\gamma}\left[\tilde{\mathbf{B}}_\gamma\right]_w(\cos\theta_\beta\partial_{x_\beta}^{-1}\partial_{\theta_\beta} - x_\beta\sin\theta_\beta - \left[\triangle\tilde{\mathbf{p}}_\beta\right]_w)\sin\theta_\alpha\partial_{x_\alpha}\Bigg]w_{\mathrm{g}}(\mathbf{x}, \boldsymbol{\theta}, t),\end{aligned} \tag{43}$$

where

$$\begin{aligned}\left[\triangle\tilde{\mathbf{p}}_\alpha\right]_w = &-\frac{e}{mc\omega}\frac{\hbar}{\mathrm{i}}\sum_{\beta,\gamma=1}^{3}\varepsilon_{\alpha\beta\gamma}\sin\theta_\beta\partial_{x_\beta}\\ &\times\int_{-1/2}^{1/2}\mathrm{d}\tau\,\tau\mathbf{B}_\gamma\left(\sin\theta_\sigma\partial_{x_\sigma}^{-1}\partial_{\theta_\sigma} + x_\sigma\cos\theta_\sigma + \frac{\mathrm{i}\hbar\tau}{m\omega}\sin\theta_\sigma\partial_{x_\sigma}, t\right),\end{aligned}$$

$$[\tilde{\mathbf{E}}]_w = \int_{-1/2}^{1/2}\mathrm{d}\tau\,\mathbf{E}\left(\sin\theta_\sigma\partial_{x_\sigma}^{-1}\partial_{\theta_\sigma} + x_\sigma\cos\theta_\sigma + \frac{\mathrm{i}\hbar\tau}{m\omega}\sin\theta_\sigma\partial_{x_\sigma}, t\right),$$

$$[\tilde{\mathbf{B}}]_w = \int_{-1/2}^{1/2}\mathrm{d}\tau\,\mathbf{B}\left(\sin\theta_\sigma\partial_{x_\sigma}^{-1}\partial_{\theta_\sigma} + x_\sigma\cos\theta_\sigma + \frac{\mathrm{i}\hbar\tau}{m\omega}\sin\theta_\sigma\partial_{x_\sigma}, t\right).$$

For symplectic tomogram $M_{\mathrm{g}}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu}, t)$ we obtain

$$\begin{aligned}\partial_t M_{\mathrm{g}}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu}, t) = \Bigg[&\frac{\boldsymbol{\mu}}{m}\partial_{\boldsymbol{\nu}} - \frac{1}{m}\sum_{\alpha=1}^{3}\left[\triangle\tilde{\mathbf{p}}_\alpha\right]_M \mu_\alpha\partial_{x_\alpha} - e\sum_{j=1}^{3}\left[\tilde{\mathbf{E}}_j\right]_M \nu_j\partial_{x_j}\\ &+ \frac{e}{mc}\sum_{\alpha,\beta,\gamma=1}^{3}\varepsilon_{\alpha\beta\gamma}\left[\tilde{\mathbf{B}}_\gamma\right]_M(\partial_{x_\beta}^{-1}\partial_{\nu_\beta} - \left[\triangle\tilde{\mathbf{p}}_\beta\right]_M)\nu_\alpha\partial_{x_\alpha}\Bigg]M_{\mathrm{g}}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu}, t),\end{aligned} \tag{44}$$

where

$$\left[\triangle\tilde{\mathbf{p}}_\alpha\right]_M = -\frac{e}{c}\frac{\hbar}{\mathrm{i}}\sum_{\beta,\gamma=1}^{3}\varepsilon_{\alpha\beta\gamma}\nu_\beta\partial_{x_\beta}\int_{-1/2}^{1/2}\mathrm{d}\tau\,\tau\mathbf{B}_\gamma(-\partial_{x_\sigma}^{-1}\partial_{\mu_\sigma} + \mathrm{i}\hbar\tau\nu_\sigma\partial_{x_\sigma}, t),$$

$$[\tilde{\mathbf{E}}]_M = \int_{-1/2}^{1/2} \mathrm{d}\tau\, \mathbf{E}\left(-\partial_{x_\sigma}^{-1}\partial_{\mu_\sigma} + \mathrm{i}\hbar\tau\nu_\sigma\partial_{x_\sigma}, t\right),$$

$$[\tilde{\mathbf{B}}]_M = \int_{-1/2}^{1/2} \mathrm{d}\tau\, \mathbf{B}\left(-\partial_{x_\sigma}^{-1}\partial_{\mu_\sigma} + \mathrm{i}\hbar\tau\nu_\sigma\partial_{x_\sigma}, t\right).$$

As it should be, equations (43) and (44) in the classical limit $\hbar \to 0$ are converted into the equations (32) and (33).

Combining formulae (29) and (41) we can write

$$M_{\mathrm{g}}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \int \langle \mathbf{q}|\hat{\rho}|\mathbf{q}'\rangle\langle \mathbf{q}'|\hat{U}_{M_{\mathrm{g}}}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu})|\mathbf{q}\rangle \mathrm{d}^3q\, \mathrm{d}^3q',$$

where we introduce the designation for the matrix element of the corresponding dequantizer

$$\langle \mathbf{q}'|\hat{U}_{M_{\mathrm{g}}}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu})|\mathbf{q}\rangle = \frac{1}{(2\pi\hbar)^3}\prod_{\sigma=1}^{3}|\nu_\sigma|^{-1}\exp\left\{\frac{\mathrm{i}}{\hbar}(q'_\sigma - q_\sigma)\left[\frac{x_\sigma}{\nu_\sigma} - \frac{\mu_\sigma(q'_\sigma + q_\sigma)}{2\nu_\sigma}\right.\right.$$
$$\left.\left. + \frac{e}{c}\int_{-1/2}^{1/2}\mathrm{d}\tau A_\sigma\left(\frac{\mathbf{q}' + \mathbf{q}}{2} + \tau(\mathbf{q}' - \mathbf{q})\right)\right]\right\}. \tag{45}$$

From (45) we can see that $\hat{U}_{M_{\mathrm{g}}}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu})$ is Hermitian and non-negative operator, consequently, the tomogram $M_{\mathrm{g}}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu})$ is real and non-negative.

From the structure of the matrix element (45) and the fact that the components of the kinetic momentum operator $\hat{\mathbf{p}} = \hat{\mathbf{P}} - \frac{e}{c}\mathbf{A}(\hat{\mathbf{q}})$ do not commute, it is possible to guess that the explicit expression for the dequantizer $\hat{U}_{M_{\mathrm{g}}}$ looks like

$$\hat{U}_{M_{\mathrm{g}}}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\exp\left\{\mathrm{i}\sum_{\sigma=1}^{3}k_\sigma\left[x_\sigma - \mu_\sigma\hat{q}_\sigma - \nu_\sigma\hat{P}_\sigma + \nu_\sigma\frac{e}{c}A_\sigma(\hat{\mathbf{q}}, t)\right]\right\}. \tag{46}$$

Indeed, calculation of the matrix element of operator (46) gives the result (45).

Formula (46) permits determining the corresponding quantizer as follows:

$$\hat{D}_{M_{\mathrm{g}}}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \left(\frac{m\omega}{2\pi}\right)^3\exp\left\{\mathrm{i}\sqrt{\frac{m\omega}{\hbar}}\sum_{\sigma=1}^{3}\left[x_\sigma - \mu_\sigma\hat{q}_\sigma - \nu_\sigma\hat{P}_\sigma + \nu_\sigma\frac{e}{c}A_\sigma(\hat{\mathbf{q}}, t)\right]\right\}. \tag{47}$$

We can see that the dequantizer and the quantizer are gauge-invariant in the sense of transformation of type (7).

After calculations for the matrix element of (47) we obtain

$$\langle \mathbf{q}|\hat{D}_{M_{\mathrm{g}}}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu})|\mathbf{q}'\rangle = \left(\frac{m\omega}{2\pi}\right)^3\exp\left\{\frac{\mathrm{i}e}{c\hbar}(\mathbf{q} - \mathbf{q}')\int_{-1/2}^{1/2}\mathrm{d}\tau\mathbf{A}\left(\frac{\mathbf{q}' + \mathbf{q}}{2} + \tau(\mathbf{q} - \mathbf{q}')\right)\right\}$$
$$\times\,\delta\left(\mathbf{q} - \mathbf{q}' - \boldsymbol{\nu}\sqrt{m\omega\hbar}\right)\exp\left\{\mathrm{i}\boldsymbol{\mu}\left[\boldsymbol{\nu}\frac{m\omega}{2} - \mathbf{q}\sqrt{\frac{m\omega}{\hbar}}\right]\right\}\prod_{\sigma=1}^{3}\exp\left\{\mathrm{i}x_\sigma\sqrt{\frac{m\omega}{\hbar}}\right\}. \tag{48}$$

Using formulae (45) and (48) it is possible to check up that

$$\int \langle \mathbf{q}_2|\hat{U}_{M_{\mathrm{g}}}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu})|\mathbf{q}_1\rangle\langle \mathbf{q}'_1|\hat{D}_{M_{\mathrm{g}}}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu})|\mathbf{q}'_2\rangle \mathrm{d}^3x\, \mathrm{d}^3\mu\, \mathrm{d}^3\nu = \delta(\mathbf{q}_1 - \mathbf{q}'_1)\delta(\mathbf{q}_2 - \mathbf{q}'_2).$$

It is obvious that the corresponding dequantizer and the quantizer for optical tomogram $w_{\mathrm{g}}(\mathbf{x}, \theta, t)$ are related with (46) and (47) as follows:

$$\hat{U}_{w_{\mathrm{g}}}(\mathbf{x}, \boldsymbol{\theta}) = \hat{U}_{M_{\mathrm{g}}}\left(x_\sigma, \mu_\sigma = \cos\theta_\sigma, \nu_\sigma = \frac{\sin\theta_\sigma}{m\omega}\right),$$

$$\hat{D}_{w_{\mathrm{g}}}(\mathbf{x}, \boldsymbol{\theta}) = \int \frac{|k_1|\,|k_2|\,|k_3|}{(m\omega)^3} \hat{D}_{M_{\mathrm{g}}}\left(k_\sigma\sqrt{\frac{\hbar}{m\omega}}x_\sigma, \mu_\sigma = k_\sigma\sqrt{\frac{\hbar}{m\omega}}\cos\theta_\sigma, \nu_\sigma = k_\sigma\sqrt{\frac{\hbar}{m\omega}}\frac{\sin\theta_\sigma}{m\omega}\right)\mathrm{d}^3k.$$

Due to the fact that the components of the operator $\hat{\mathbf{x}}(\boldsymbol{\theta})$ as well as $\hat{\mathbf{x}}(\boldsymbol{\mu}, \boldsymbol{\nu})$ do not commute, the tomographic representations constructed in this section are not probability representations, but are non-negative, normalized, and gauge-independent quasi-probability tomographic representations.

5. Gauge-independent probability representation

Unfortunately, the gauge-independent tomographic functions introduced in the previous section $M_{\mathrm{g}}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu}, t)$ and $w_{\mathrm{g}}(\mathbf{x}, \boldsymbol{\theta}, t)$ are not distribution functions of any physical observable. To make up for this shortcoming, we introduce the tomographic function $\mathfrak{M}(x, \boldsymbol{\mu}, \boldsymbol{\nu}, t)$ as the following map of the gauge-independent Wigner function:

$$\mathfrak{M}(x, \boldsymbol{\mu}, \boldsymbol{\nu}, t) = \int W_{\mathrm{g}}(\mathbf{q}, \mathbf{p}, t)\delta(x - \boldsymbol{\mu}\mathbf{q} - \boldsymbol{\nu}\mathbf{p})\mathrm{d}^3q\,\mathrm{d}^3p. \tag{49}$$

Such a map was applied for the construction of center of mass tomography [39].

It is evident that $\mathfrak{M}(x, \boldsymbol{\mu}, \boldsymbol{\nu}, t)$ is a distribution function of the physical observable $\hat{x}(\boldsymbol{\mu}, \boldsymbol{\nu}) = \boldsymbol{\mu}\hat{\mathbf{q}} + \boldsymbol{\nu}\hat{\mathbf{p}}$, which is a scalar product of two 6-dimensional vectors $(\boldsymbol{\mu},\ \boldsymbol{\nu})$ and $(\hat{\mathbf{q}},\ \hat{\mathbf{p}})$. The quantity $\mathfrak{M}(x, \boldsymbol{\mu}, \boldsymbol{\nu}, t)\mathrm{d}x$ is the probability to have the value of the scalar operator $\hat{x}(\boldsymbol{\mu}, \boldsymbol{\nu})$ within the interval between x and $x + \mathrm{d}x$ at fixed time t and fixed vector $(\boldsymbol{\mu},\ \boldsymbol{\nu})$.

The map inverse to (49) has, obviously, the form:

$$W_{\mathrm{g}}(\mathbf{q}, \mathbf{p}, t) = \left(\frac{m\omega}{4\pi^2\hbar}\right)^3 \int \mathfrak{M}(x, \boldsymbol{\mu}, \boldsymbol{\nu}, t)\exp\left\{\mathrm{i}\sqrt{\frac{m\omega}{\hbar}}(x - \boldsymbol{\mu}\mathbf{q} - \boldsymbol{\nu}\mathbf{p})\right\}\mathrm{d}x\,\mathrm{d}^3\mu\,\mathrm{d}^3\nu. \tag{50}$$

Combining formulae (49) and (41) we obtain the expression for the matrix element of the dequantizer operator $\hat{U}_{\mathfrak{M}}(x, \boldsymbol{\mu}, \boldsymbol{\nu})$ for this representation

$$\begin{aligned}\langle\mathbf{q}'|\hat{U}_{\mathfrak{M}}(x, \boldsymbol{\mu}, \boldsymbol{\nu})|\mathbf{q}\rangle = {} & \frac{1}{2\pi\hbar|\nu_3|}\delta\left(\frac{\nu_1}{\nu_3}(q_3' - q_3) - (q_1' - q_1)\right)\delta\left(\frac{\nu_2}{\nu_3}(q_3' - q_3) - (q_2' - q_2)\right) \\ & \times \exp\left\{\frac{\mathrm{i}}{\hbar}\left[x\frac{q_3' - q_3}{\nu_3} - \mu_1\frac{q_1'^2 - q_1^2}{2\nu_1} - \mu_2\frac{q_2'^2 - q_2^2}{2\nu_2} - \mu_3\frac{q_3'^2 - q_3^2}{2\nu_3}\right]\right\} \\ & \times \exp\left\{\frac{\mathrm{i}e}{c\hbar}(\mathbf{q}' - \mathbf{q})\int_{-1/2}^{1/2}\mathrm{d}\tau\mathbf{A}\left(\frac{\mathbf{q}' + \mathbf{q}}{2} + \tau(\mathbf{q}' - \mathbf{q})\right)\right\}.\end{aligned} \tag{51}$$

Taking into account expressions of dequantizer operators in the previous sections and the matrix element (51) we can write the explicit expression for the gauge-invariant dequantizer $\hat{U}_{\mathfrak{M}}(x, \boldsymbol{\mu}, \boldsymbol{\nu})$

$$\hat{U}_{\mathfrak{M}}(x,\boldsymbol{\mu},\boldsymbol{\nu}) = \int \frac{\mathrm{d}k}{2\pi} \exp\left\{\mathrm{i}k\left[x - \boldsymbol{\mu}\hat{\mathbf{q}} - \boldsymbol{\nu}\hat{\mathbf{P}} + \frac{e}{c}\boldsymbol{\nu}\mathbf{A}(\hat{\mathbf{q}})\right]\right\}. \tag{52}$$

Then the quantizer $D_{\mathfrak{M}}(x,\boldsymbol{\mu},\boldsymbol{\nu})$, obviously, equals

$$\hat{D}_{\mathfrak{M}}(x,\boldsymbol{\mu},\boldsymbol{\nu}) = \left(\frac{m\omega}{2\pi}\right)^3 \exp\left\{\mathrm{i}\sqrt{\frac{m\omega}{\hbar}}\left[x - \boldsymbol{\mu}\hat{\mathbf{q}} - \boldsymbol{\nu}\hat{\mathbf{P}} + \frac{e}{c}\boldsymbol{\nu}\mathbf{A}(\hat{\mathbf{q}})\right]\right\}, \tag{53}$$

and the calculation of its matrix element gives rise to the following:

$$\begin{aligned}\langle\mathbf{q}|\hat{D}_{\mathfrak{M}}(x,\boldsymbol{\mu},\boldsymbol{\nu})|\mathbf{q}'\rangle = \left(\frac{m\omega}{2\pi}\right)^3 \exp\left\{\frac{\mathrm{i}e}{c\hbar}(\mathbf{q}-\mathbf{q}')\int_{-1/2}^{1/2}\mathrm{d}\tau\,\mathbf{A}\left(\frac{\mathbf{q}'+\mathbf{q}}{2} + \tau(\mathbf{q}-\mathbf{q}')\right)\right\}\\ \times\delta\left(\mathbf{q}-\mathbf{q}'-\boldsymbol{\nu}\sqrt{m\omega\hbar}\right)\exp\left\{\mathrm{i}\boldsymbol{\mu}\left[\boldsymbol{\nu}\frac{m\omega}{2} - \mathbf{q}\sqrt{\frac{m\omega}{\hbar}}\right] + \mathrm{i}x\sqrt{\frac{m\omega}{\hbar}}\right\}.\end{aligned} \tag{54}$$

The correspondence rules (30) for representation $\mathfrak{M}(x,\boldsymbol{\mu},\boldsymbol{\nu})$ acquire a slightly modernized form

$$\begin{aligned} q_\sigma W_{\mathrm{g}}(\mathbf{q},\mathbf{p}) &\leftrightarrow -\partial_x^{-1}\partial_{\mu_\sigma}\mathfrak{M}(x,\boldsymbol{\mu},\boldsymbol{\nu}),\\ p_\sigma W_{\mathrm{g}}(\mathbf{q},\mathbf{p}) &\leftrightarrow -\partial_x^{-1}\partial_{\nu_\sigma}\mathfrak{M}(x,\boldsymbol{\mu},\boldsymbol{\nu}),\\ \partial_{q_\sigma} W_{\mathrm{g}}(\mathbf{q},\mathbf{p}) &\leftrightarrow \mu_\sigma\partial_x\mathfrak{M}(x,\boldsymbol{\mu},\boldsymbol{\nu}),\\ \partial_{p_\sigma} W_{\mathrm{g}}(\mathbf{q},\mathbf{p}) &\leftrightarrow \nu_\sigma\partial_x\mathfrak{M}(x,\boldsymbol{\mu},\boldsymbol{\nu}).\end{aligned} \tag{55}$$

With the help of (55) equation (42) is transformed to the evolution equation for the tomogram $\mathfrak{M}$

$$\begin{aligned}\partial_t\mathfrak{M}(x,\boldsymbol{\mu},\boldsymbol{\nu},t) = \Bigg[\frac{\boldsymbol{\mu}}{m}\partial_{\boldsymbol{\nu}} - \frac{1}{m}\sum_{\alpha=1}^{3}\left[\triangle\tilde{\mathbf{p}}_\alpha\right]_{\mathfrak{M}}\mu_\alpha\partial_x - e\sum_{j=1}^{3}\left[\tilde{\mathbf{E}}_j\right]_{\mathfrak{M}}\nu_j\partial_x\\ + \frac{e}{mc}\sum_{\alpha,\beta,\gamma=1}^{3}\varepsilon_{\alpha\beta\gamma}\left[\tilde{\mathbf{B}}_\gamma\right]_{\mathfrak{M}}\left(\partial_{\nu_\beta} - \left[\triangle\tilde{\mathbf{p}}_\beta\right]_{\mathfrak{M}}\partial_x\right)\nu_\alpha\Bigg]\mathfrak{M}(x,\boldsymbol{\mu},\boldsymbol{\nu},t),\end{aligned} \tag{56}$$

where

$$\left[\triangle\tilde{\mathbf{p}}_\alpha\right]_M = -\frac{e}{c}\frac{\hbar}{\mathrm{i}}\sum_{\beta,\gamma=1}^{3}\varepsilon_{\alpha\beta\gamma}\nu_\beta\partial_x\int_{-1/2}^{1/2}\mathrm{d}\tau\,\tau\mathbf{B}_\gamma(-\partial_x^{-1}\partial_{\mu_\sigma} + \mathrm{i}\hbar\tau\nu_\sigma\partial_x, t),$$

$$[\tilde{\mathbf{E}}]_M = \int_{-1/2}^{1/2}\mathrm{d}\tau\,\mathbf{E}\left(-\partial_x^{-1}\partial_{\mu_\sigma} + \mathrm{i}\hbar\tau\nu_\sigma\partial_x, t\right),$$

$$[\tilde{\mathbf{B}}]_M = \int_{-1/2}^{1/2}\mathrm{d}\tau\,\mathbf{B}\left(-\partial_x^{-1}\partial_{\mu_\sigma} + \mathrm{i}\hbar\tau\nu_\sigma\partial_x, t\right).$$

In the limit case $\hbar\to 0$ we get the classical equation

$$\partial_t \mathfrak{M}_{\mathrm{cl}}(x, \boldsymbol{\mu}, \boldsymbol{\nu}, t) = \left[\frac{\boldsymbol{\mu}}{m}\partial_{\boldsymbol{\nu}} + \frac{e}{mc}\sum_{\alpha,\beta,\gamma=1}^{3} \varepsilon_{\alpha\beta\gamma} B_\gamma(-\partial_x^{-1}\partial_{\mu_\sigma}, t)\nu_\alpha \partial_{\nu_\beta} - e\sum_{j=1}^{3} E_j(-\partial_x^{-1}\partial_{\mu_\sigma}, t)\nu_j \partial_x\right]\mathfrak{M}_{\mathrm{cl}}(x, \boldsymbol{\mu}, \boldsymbol{\nu}, t), \tag{57}$$

which is the Liouville equation in the corresponding representation. Thus, we have constructed the gauge-independent probability representation which has a clear physical meaning and the classical limit.

The density matrix $\rho(\mathbf{q}, \mathbf{q}', t)$ depends on time and on six spatial variables, while the tomogram $\mathfrak{M}(x, \boldsymbol{\mu}, \boldsymbol{\nu}, t)$ depends on time, on one spatial variable, and on six tomography parameters. But the number of these parameters can be reduced by one if we take into account that the tomogram $\mathfrak{M}(x, \boldsymbol{\mu}, \boldsymbol{\nu}, t)$ is a homogeneous function in the sense

$$\mathfrak{M}(rx, r\boldsymbol{\mu}, r\boldsymbol{\nu}, t) = |r|^{-1}\mathfrak{M}(x, \boldsymbol{\mu}, \boldsymbol{\nu}, t).$$

Therefore, in the 6-dimensional space $(\boldsymbol{\mu}, \tilde{\boldsymbol{\nu}}) = (\boldsymbol{\mu}, m\omega\boldsymbol{\nu})$ one can, for instance, pass to the unit sphere and reduce the number of variables introducing the new tomogram $\mathfrak{w}(x, \boldsymbol{\xi}, t)$ as follows:

$$\mathfrak{w}(x, \boldsymbol{\xi}, t) = \mathfrak{M}\left(x, \boldsymbol{\mu}(\boldsymbol{\xi}), \frac{\tilde{\boldsymbol{\nu}}(\boldsymbol{\xi})}{m\omega}, t\right) = \int W_{\mathrm{g}}(\mathbf{q}, \mathbf{p}, t)\delta\left(x - \boldsymbol{\mu}(\boldsymbol{\xi})\,\mathbf{q} - \tilde{\boldsymbol{\nu}}(\boldsymbol{\xi})\,\frac{\mathbf{p}}{m\omega}\right)\mathrm{d}^3q\,\mathrm{d}^3p, \tag{58}$$

where $\boldsymbol{\xi}$ is a 5-dimensional vector of directional angles in the 6-dimensional space and

$$\begin{pmatrix}\boldsymbol{\mu}(\boldsymbol{\xi})\\ \tilde{\boldsymbol{\nu}}(\boldsymbol{\xi})\end{pmatrix} = \begin{pmatrix}\mu_1(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)\\ \mu_2(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)\\ \mu_3(\xi_2, \xi_3, \xi_4, \xi_5)\\ \nu_1(\xi_3, \xi_4, \xi_5)\\ \nu_2(\xi_4, \xi_5)\\ \nu_3(\xi_5)\end{pmatrix} = \begin{pmatrix}\sin\xi_1\sin\xi_2\sin\xi_3\sin\xi_4\sin\xi_5\\ \cos\xi_1\sin\xi_2\sin\xi_3\sin\xi_4\sin\xi_5\\ \cos\xi_2\sin\xi_3\sin\xi_4\sin\xi_5\\ \cos\xi_3\sin\xi_4\sin\xi_5\\ \cos\xi_4\sin\xi_5\\ \cos\xi_5\end{pmatrix}. \tag{59}$$

In the physical meaning $\mathfrak{w}(x, \boldsymbol{\xi}, t)\mathrm{d}x$ is the probability of the system to have the projection of the vector $(\mathbf{q},\ \mathbf{p}/m\omega)$ on the direction of the unit vector (59) within the interval between x and $x + \mathrm{d}x$.

The inverse transformation $\mathfrak{w}(x, \boldsymbol{\xi}, t) \to W_{\mathrm{g}}(\mathbf{q}, \mathbf{p}, t)$ has, obviously, the form:

$$W_{\mathrm{g}}(\mathbf{q}, \mathbf{p}, t) = (4\pi^2 m\omega)^{-3}\int \mathfrak{w}(x, \boldsymbol{\xi}, t)\exp\left\{\mathrm{i}r\left(x - \boldsymbol{\mu}(\boldsymbol{\xi})\mathbf{q} - \tilde{\boldsymbol{\nu}}(\boldsymbol{\xi})\frac{\mathbf{p}}{m\omega}\right)\right\} \times r^5\sin\xi_2\sin^2\xi_3\sin^3\xi_4\sin^4\xi_5\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}^5\xi. \tag{60}$$

So, the tomogram $\mathfrak{w}(x, \boldsymbol{\xi}, t)$ also contains all available information about the state of the system under study, but it depends on the same number of variables as the density matrix, and it is gauge-independent.

For the function $\mathfrak{w}(x, \boldsymbol{\xi}, t)$ it is also possible to write the evolution equation, and it is possible to reduce the number of variables of $\mathfrak{M}(x, \boldsymbol{\mu}, \boldsymbol{\nu}, t)$ by a more symmetrical method different from (58) and (59), but it may be the subject of future publications.

6. Conclusion

In conclusion, we point out that the evolution equation of a tomogram of the state of quantum system, as well as the appropriate Moyal equation, possess a gauge invariance. However the optical and symplectic tomograms in their determination with the help of gauge-independent dequantizers (15) and (19) do not possess the gauge independence and are converted by the integral transformation (14) with the kernel of type (13) dependent on the quantizer and dequantizer operators, and the gauge function χ.

Contrary to the quantum case, optical and symplectic tomograms of classical distribution function in the phase space with kinetic momentum possess of the gauge independence. Therefore, in the electro-magnetic field the evolution equations (36) and (39) for gauge-independent tomograms do not have the classical limit (32) and (33) when $\hbar \to 0$. This quality differs from the quality of the Moyal equation, which is gauge-dependent but, nevertheless, has the gauge-independent Liouville equation as the classical limit.

To solve this problem we introduced the gauge-independent optical and symplectic tomographic quasi-distributions and tomographic probability distributions, and obtained their gauge-independent evolution equations, which are converted in the classical limit to the Liouville equation in corresponding tomographic representations.

We pointed out that the motivation to study the gauge invariance in tomographic probability representation is closely related with studies of gauge invariance in Wigner-Weyl representation. We have shown that gauge-independent tomograms can be constructed and corresponding quantum evolution equations can be obtained.

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References

[1] Landau L D and Lifshitz E M 1997 *Field Theory* (Oxford: Pergamon)
[2] Berestetskii V B, Lifshitz E M and Pitaevskii L P 1982 *Quantum Electrodinamics* (Oxford: Pergamon)
[3] Yang C N and Mills R 1954 *Phys. Rev.* **96** 191
[4] Landau L D and Lifshitz E M 1997 *Quantum Mechanics* (Oxford: Pergamon)
[5] Schrodinger E 1926 *Ann. Phys.* **79** 361
[6] Wigner E 1932 *Phys. Rev.* **40** 749
[7] Blohintsev D I 1940 *J. Phys. USSA* **2** 71
[8] Glauber R J 1963 *Phys. Rev. Lett.* **10** 84
[9] Sudarshan E C G 1963 *Phys. Rev. Lett.* **10** 277
[10] Husimi K 1940 *Proc. Phys. Math. Soc. Japan* **22** 264
[11] Klimontovich Iu L and Silin V P 1952 *J. Exper. Theoret. Phys. USSR* **23** 151
[12] Silin V P 1952 *J. Exp. Theor. Phys. USSR* **23** 641
[13] Silin V P 1954 *J. Exp. Theor. Phys. USSR* **27** 269
[14] Silin V P 1955 *J. Exp. Theor. Phys. USSR* **28** 749
Silin V P 1955 *Sov. Phys.—JETP* **1** 607 (Engl. transl.)
[15] Landau L D 1957 *J. Exp. Theor. Phys. USSR* **32** 59
Landau L D 1957 *Sov. Phys.—JETP* **5** 101 (Engl. transl.)
[16] Mancini S, Man'ko V I and Tombesi P 1996 *Phys. Lett.* A **213** 1
[17] Ibort A, Man'ko V I, Marmo G, Simoni A and Ventriglia F 2009 *Phys. Scr.* **79** 065013
[18] Bertrand J and Bertrand P 1987 *Found. Phys.* **17** 397

[19] Vogel K and Risken H 1989 *Phys. Rev.* A **40** 2847
[20] Mancini S, Man'ko V I and Tombesi P 1995 *J. Opt. B: Quantum Semiclass. Opt.* **7** 615
[21] Man'ko V I and Man'ko O V 1997 *J. Exp. Theor. Phys.* **85** 430
[22] Dodonov V V and Man'ko V I 1997 *Phys. Lett.* A **229** 335
[23] Man'ko O V, Man'ko V I and Marmo G 2002 *J. Phys. A: Math. Gen.* **35** 699
[24] Man'ko O V, Man'ko V I, Marmo G and Vitale P 2007 *Phys. Lett.* A **360** 522
[25] Stratonovich R L 1957 *J. Exper. Theoret. Phys. USSR* **31** 1012
Stratonovich R L 1957 *Sov. Phys.—JETP* **4** 891 (Engl. transl.)
[26] Lizzi F and Vitale P 2014 *SIGMA* **10** 086
[27] Zachos C K, Fairlie D B and Curtrigh T L 2005 *Quantum Mechanics in Phase Space: an Overview with Selected Papers* (Singapore: World Scientific)
[28] Korennoy Ya A and Man'ko V I 2011 *J. Russ. Laser Res.* **32** 74
[29] Stratonovich R L 1956 *Dokl. Akad. Nauk SSSR* **1** 72
Stratonovich R L 1956 *Sov. Phys.—Dokl.* **1** 414 (Engl. transl.)
[30] Serimaa O T, Javanainen J and Varró S 1986 *Phys. Rev.* A **33** 2913
[31] Javanainen J, Varró S and Serimaa O T 1987 *Phys. Rev.* A **35** 2791
[32] Varró S and Javanainen J 2003 *J. Opt. B: Quantum Semiclass. Opt.* **5** S402
[33] Nedjalkov M, Weinbub J, Ellinghaus P and Selberherr S 2015 *J. Comput. Electron.* **14** 888
[34] Korennoy Ya A and Man'ko V I 2011 *J. Russ. Laser Res.* **32** 338
[35] Amosov G G, Korennoy Ya A and Man'ko V I 2012 *Phys. Rev.* A **85** 052119
[36] Korennoy Ya A and Man'ko V I 2015 *J. Russ. Laser Res.* **36** 534
[37] Korennoy Ya A and Man'ko V I 2016 *Int. J. Theor. Phys.* **55** 4885
[38] Moyal J E 1949 *Proc. Cambrige Phil. Soc.* **45** 99
[39] Arkhipov A S, Lozovik Y E and Man'ko V I 2004 *Phys. Lett.* A **328** 419