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Tomographic quantum measures for many degrees of freedom and the central limit theorem

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Abstract
A tomographic quantum measure for a multimode system is introduced. Symplectic tomograms describing quantum states of the system with many degrees of freedom are shown to be equal to partial derivatives of the von Neumann probability distribution functions of homodyne random variables. The central limit theorem known in quantum probability theory is applied to describe properties of the symplectic quantum measures introduced. An example of the centre-of-mass homodyne quadrature is studied in the context of the central limit theorem.

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1. Introduction

Probabilistic aspects of quantum tomograms by means of the von Neumann approach [1] were considered in [2–4], where a connection of quantum tomograms with quantum probability measures was shown. In quantum mechanics, one describes the quantum states either by a vector $|\psi>$ in a Hilbert space (for pure states) or by a density operator $\hat{\rho}$ (for mixed states). There exist several different equivalent representations of quantum states and quantum observables ([5]). Quantum tomographic probability distributions (called tomograms of the quantum states) contain the same information on the states as the density operators. Given the state tomogram, one can calculate all the physical characteristics of the system (e.g., of a chain of trapped ions considered as elements of a possible quantum computing device [6]). It means that the standard positive probability distribution can be used as an alternative to the wavefunction or density matrix for describing the quantum state. Note that in [9–11] the tomographic probability distributions were found to be related to Wigner functions [12]. The tomographic probability distribution of [9, 10] was used in optical tomography scheme [13, 14] to reconstruct the Wigner function of photon states by measuring the homodyne quadrature distributions and applying the Radon transform [15] to find the Wigner function. In [11], an extension of optical tomography to symplectic tomography [16] was suggested. In the framework of the symplectic tomography scheme, the Wigner function and density operator can be reconstructed using a Fourier-like integral (connected with the Radon integral) of the symplectic tomogram. Recently centre-of-mass tomography was introduced ([17]). Here we investigate its properties in the context of the central limit theorem. Some brief remarks on potential advantages of the tomographic probability representation in using fundamental results of probability theory (central limit theorem) to study quantum states were mentioned in [7, 8]. The aim of this work is to obtain in explicit form the manifestation of the central limit theorem in the tomographic probability representation.

This paper is organized as follows. In section 2, we point out which probability distributions can be associated with quantum observables. In section 3, the tomographic approach for a multimode system is described. It is shown that the centre-of-mass tomogram has a Gaussian distribution under certain conditions. In section 4, we show how the central limit theorem works for the Fock states.

2. Probability distributions associated with quantum observables

Denote by $\mathcal{L}(H)$ and $\sigma(H)$ the set of all Hermitian operators (quantum observables) and the set of states (positive unit-trace operators) in a separable Hilbert space H, respectively. Given $\hat{x} \in \mathcal{L}(H)$, there exists an orthogonal projection-valued measure $\mathrm{d}\hat{M}(X)$ on the real line $\mathbb{R}$ such that $\hat{x} = \int_{\mathbb{R}} X \mathrm{d}\hat{M}(X)$. The measure $\mathrm{d}\hat{M}(X)$ satisfies the property $\hat{M}(\Omega_1)$ and $\hat{M}(\Omega_2)$ are pairwise orthogonal projections for any two disjoint Borel subsets $\Omega_1, \Omega_2 \subset \mathbb{R}$. If $\hat{x}_1, \ldots, \hat{x}_n \in \mathcal{L}(H)$ are pairwise commuting observables, then their spectral measures $\mathrm{d}\hat{M}_1, \ldots, \mathrm{d}\hat{M}_n$ are commuting. Given a state $\hat{\rho} \in \sigma(H)$ one can define the joint probability distribution associated with commuting observables $\hat{x}_1, \ldots, \hat{x}_n$ as follows (see [1, 18]):

$$P(\hat{x}_1 \in \Omega_1, \ldots, \hat{x}_n \in \Omega_n) = \mathrm{Tr}(\hat{\rho}\hat{M}_1(\Omega_1) \ldots \hat{M}_n(\Omega_n)), \tag{1}$$

where Ω_k are Borel subsets of $\mathbb{R}$. If $n = 1$, we shall call (1) by a probability distribution of the observable $\hat{x}_1 \equiv \hat{x}$ (in the state ρ). Then, the expectation and variance of $\hat{x}$ are given by the formulae

$$\mathbb{E}(\hat{x}) = \mathrm{Tr}(\hat{\rho}\hat{x}),$$
$$\mathrm{Var}(\hat{x}) = \mathrm{Tr}(\hat{\rho}\hat{x}^2) - (\mathrm{Tr}(\hat{\rho}\hat{x}))^2.$$

Let us involve the distribution function of (1) as

$$\mathcal{M}_{\hat{\rho}}^{\hat{x}_1,\ldots,\hat{x}_n}(X_1, \ldots, X_n) \equiv \mathcal{M}_{\hat{\rho}}(X_1, \ldots, X_n)$$
$$= P(\hat{x}_1 \in (-\infty, X_1], \ldots, \hat{x}_n \in (-\infty, X_n]).$$

Commuting observables $\hat{x}_1, \ldots, \hat{x}_N$ are said to be (boson) independent if their joint probability distributions (1) satisfy the classical independent property,

$$P(\hat{x}_{i_1} \in \Omega_1, \ldots, \hat{x}_{i_n} \in \Omega_n) = P(\hat{x}_{i_1} \in \Omega_1) \ldots P\big(\hat{x}_{i_n} \in \Omega_n\big),$$

for any $1 \leqslant i_1, \ldots, i_n \leqslant N, i_s \neq i_l, s \neq l$, and any Borel subsets $\Omega_k \subset \mathbb{R}$. For commuting independent observables $\hat{x}_1, \ldots, \hat{x}_N, \ldots$ with the finite third moments $m_k = \mathrm{Tr}(|\hat{x}_k - \mathbb{E}(\hat{x}_k)|^3)$ satisfying the property

$$\lim_{N\to+\infty} \frac{\sum_{k=1}^N m_k}{\left(\sum_{k=1}^N \mathrm{Var}(\hat{x}_k)\right)^{3/2}} = 0,$$

the central limit theorem holds (see [18]), that is the probability distribution of the sums $\hat{s}_N = \hat{x}_1 + \cdots + \hat{x}_N$ goes to Gaussian one with the expectation $\mathbb{E}(\hat{s}_N) = \sum_{k=1}^{N} \mathbb{E}(\hat{x}_k)$ and the variance $\mathrm{Var}(\hat{s}_N) = \sum_{k=1}^{N} \mathrm{Var}(\hat{x}_k)$ as $N \to +\infty$.

3. Quantum tomograms for multimode system

Consider a multimode system characterized by the position and momentum operators $\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_n, \hat{p}_n$. We suppose that $\hat{q}_j, \hat{p}_j$ are commuting for different indices j. Denote $\overline{X} = (X_1, \ldots, X_n)$, $\overline{\mu} = (\mu_1, \ldots, \mu_n)$, $\overline{\nu} = (\nu_1, \ldots, \nu_n)$ arbitrary collections of real numbers. The quantum tomogram determining a state $\hat{\rho}$ of the multimode system is defined by the formula

$$\omega(\overline{X}, \overline{\mu}, \overline{\nu}) = \mathrm{Tr}(\hat{\rho}\delta(X_1 - \mu_1\hat{q}_1 - \nu_1\hat{p}_1)\ldots\delta(X_n - \mu_n\hat{q}_n - \nu_n\hat{p}_n)),$$

where δ is the operator-valued Dirac δ-functions. Put $\hat{x}_1 = \mu_1\hat{q}_1 + \nu_1\hat{p}_1, \ldots, \hat{x}_n = \mu_n\hat{q}_n + \nu_n\hat{p}_n$. Then, $\hat{x}_1, \ldots, \hat{x}_n$ are commuting observables and

$$\omega(\overline{X}, \overline{\mu}, \overline{\nu}) = \frac{\partial^n}{\partial X_1 \cdots \partial X_n} \mathcal{M}_{\hat{\rho}}^{\hat{x}_1,\ldots,\hat{x}_n}(X_1, \ldots, X_n). \tag{2}$$

It is possible to reconstruct the state $\hat{\rho}$ from the tomogram (2) as follows:

$$\hat{\rho} = \frac{1}{(2\pi)^n} \int \omega(\overline{X}, \overline{\mu}, \overline{\nu})\, \mathrm{e}^{\mathrm{i}(\overline{X} - \overline{\mu}\hat{\overline{q}} - \overline{\nu}\hat{\overline{p}})}\, \mathrm{d}\overline{X}\, \mathrm{d}\overline{\mu}\, \mathrm{d}\overline{\nu}$$

$$= \frac{1}{(2\pi)^n} \int \mathrm{e}^{\mathrm{i}(\overline{X} - \overline{\mu}\hat{\overline{q}} - \overline{\nu}\hat{\overline{p}})}\, \mathrm{d}\mathcal{M}_{\hat{\rho}}^{\hat{x}_1,\ldots,\hat{x}_n}(\overline{X})\, \mathrm{d}\overline{\mu}\, \mathrm{d}\overline{\nu}.$$

The alternative approach to the description of quantum states in a multimode system is a centre-of-mass tomogram defined as

$$\omega_{\mathrm{cm}}(X, \overline{\mu}, \overline{\nu}) = \mathrm{Tr}(\hat{\rho}\delta(X - \overline{\mu}\hat{\overline{q}} - \overline{\nu}\hat{\overline{p}})), \tag{3}$$

where $\overline{a}\overline{b}$ denotes a scalar product of vectors $\overline{a}$ and $\overline{b}$. The state $\hat{\rho}$ can be reconstructed from the tomogram ω_{cm} as follows:

$$\hat{\rho} = \int \omega_{\mathrm{cm}}(X, \overline{\mu}, \overline{\nu})\, \mathrm{e}^{\mathrm{i}(X - \overline{\mu}\hat{\overline{q}} - \overline{\nu}\hat{\overline{p}})} \frac{\mathrm{d}X\, \mathrm{d}\overline{\mu}\, \mathrm{d}\overline{\nu}}{(2\pi)^n}.$$

There is a connection of the Wigner function [12] with the tomogram ω_{cm} (see [17]). The position of the centre-of-mass for n particles measured in a scaled and rotated reference frame of the phase space is given by the observable

$$\hat{x}_{\mathrm{cm}} = \sum_{i=1}^{n} \frac{M_i(\mu_i\hat{q}_i + \nu_i\hat{p}_i)}{M}, \tag{4}$$

where M_i and $M = \sum_{i=1}^{n} M_{\mathrm{i}}$ are the mass of the ith particle and the total mass, respectively. It follows that ω_{cm} is a tomogram of the centre-of-mass. Suppose that there exist $N + 1$ numbers j_s, $1 = j_0 < j_1 < j_2 < \cdots < j_N = n$, such that the observables $\hat{x}_k = \sum_{s=j_{k-1}}^{j_k-1} (\mu_s\hat{q}_s + \nu_s\hat{p}_s)$, $1 \leqslant k \leqslant N$, are independent and satisfy the condition of the central limit theorem. Note that $\sum_{k=1}^{N} \hat{x}_k = \overline{\mu}\hat{\overline{q}} + \overline{\nu}\hat{\overline{p}} \equiv \hat{s}_N$. The centre-of-mass tomogram (3) is a density of probability distribution associated with the observable $\hat{s}_N$. Hence, the central limit theorem results in the Gaussian distribution of ω_{cm} under the conditions we impose.

4. The centre-of-mass tomography for the Fock states

For any vector $\overline{n}$ whose components are positive integer numbers n_k, $1 \leqslant k \leqslant N$, the wavefunction

$$\psi^{\overline{n}}(\overline{X}) = \prod_{k=1}^{N} \frac{\mathrm{e}^{-\frac{X_k^2}{2}}}{\pi^{1/4}\sqrt{2^{n_k} n_k!}} H_{n_k}(X_k)$$

determines the multimode Fock state. Here H_{n_k} are Hermite polynomials. The symplectic tomogram ω of the Fock state is given by the formula

$$\omega(\overline{X}, \overline{\mu}, \overline{\nu}) = \prod_{k=1}^{N} \frac{\mathrm{e}^{-\frac{X_k^2}{\mu_k^2+\nu_k^2}}}{2^{n_k} n_k! \sqrt{\pi\left(\mu_k^2+\nu_k^2\right)}} H_{n_k}^2\left(\frac{X_k^2}{\sqrt{\mu_k^2+\nu_k^2}}\right).$$

We will consider below another kind of tomography which is the centre-of-mass tomography of quantum states. In [17], the formal expression was found for the centre-of-mass tomogram of the Fock state which has the form

$$\omega_{\mathrm{cm}}(X, \overline{\mu}, \overline{\nu}) = \int \delta\left(X - \sum_{k=1}^{N} X_k\right) \prod_{k=1}^{N} \frac{\mathrm{e}^{-\frac{X_k^2}{\mu_k^2+\nu_k^2}}}{2^{n_k} n_k! \sqrt{\pi\left(\mu_k^2+\nu_k^2\right)}} H_{n_k}^2\left(\frac{X_k^2}{\sqrt{\mu_k^2+\nu_k^2}}\right) \mathrm{d}\overline{X}.$$

The expectations and variances of the variables $\hat{x}_k = \mu_k \hat{q}_k + \nu_k \hat{p}_k$ in the Fock state are

$$\mathbb{E}(\hat{x}_k) = 0,$$
$$\mathrm{Var}(\hat{x}_k) = \left(\tfrac{1}{2} + n_k\right)\left(\mu_k^2+\nu_k^2\right),$$

$1 \leqslant k \leqslant N$. Suppose that $N \to +\infty$ and the components of $\overline{n}$ are uniformly bounded such that $n_k \leqslant n$. Then the absolute third moments of $\hat{x}_k$ are uniformly bounded by means of

$$m_k = \mathrm{Tr}(|\hat{x}_k|^3) = 2\int_0^{+\infty} X^3 \frac{\mathrm{e}^{-\frac{X^2}{\mu_k^2+\nu_k^2}}}{2^{n_k} n_k! \sqrt{\pi\left(\mu_k^2+\nu_k^2\right)}} H_{n_k}^2\left(\frac{X}{\sqrt{\mu_k^2+\nu_k^2}}\right) \mathrm{d}X \leqslant C\left(\mu_k^2+\nu_k^2\right)^{3/2},$$

where the constant C does not depend on μ_k, ν_k and k. Testing the known condition of applicability of the central limit theorem gives us

$$\frac{\sum_{k=1}^{N} m_k}{\left(\sum_{k=1}^{N} \mathrm{Var}(\hat{x}_k)^2\right)^{3/2}} \leqslant \frac{C\sum_{k=1}^{N}\left(\mu_k^2+\nu_k^2\right)^{3/2}}{\left(\sum\left(\frac{1}{2}+n_k\right)\left(\mu_k^2+\nu_k^2\right)\right)^{3/2}} \equiv S_N.$$

Suppose that $0 < r < \mu_k^2 + \nu_k^2 < R < +\infty$, then $S_N \to 0$ as $N \to +\infty$. The application of the central limit theorem implies that the centre-of-mass tomogram ω_{cm} of the Fock state tends to a density of Gaussian distribution with the expectation and variance equal to zero and $\sum_{k=1}^{N}\left(\frac{1}{2}+n_k\right)\left(\mu_k^2+\nu_k^2\right)$, respectively.

5. Conclusion

We studied the probability characteristics associated with a multimode quantum system. It is shown that the centre-of-mass tomogram of the Fock state is asymptotically Gaussian if the number of modes N tends to infinity. This result proves the intuitively obvious statement that for a many-body system (macroscopic one), the centre-of-mass is coordinate is practically described by a Gaussian density matrix independently on the characteristics of density matrices of the microscopic subsystems of the combined system.

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